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Towards gauge theory for a class of commutative and non-associative fuzzy spaces

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Abstract: We discuss gauge theories for commutative but non-associative algebras related to the SO$(2k+1)$ covariant finite dimensional fuzzy $2k$-sphere algebras. A consequence of non-associativity is that gauge fields and gauge parameters have to be generalized to be functions of coordinates as well as derivatives. The usual gauge fields depending on coordinates only are recovered after a partial gauge fixing. The deformation parameter for these commutative but non-associative algebras is a scalar of the rotation group. This suggests interesting string-inspired algebraic deformations of spacetime which preserve Lorentz-invariance.

Keywords: D-branes, M(atrix) Theories, Space-Time Symmetries, Gauge Symmetry.

Contents

1. Introduction

The fuzzy four-sphere [1] has several applications in D-brane physics [2, 3]. While the fuzzy four-sphere has some similarities to the fuzzy two-sphere [4], it is also different in important ways. The Matrix Algebra of the fuzzy $2k$-sphere at finite n is the space of transformations, $\mathrm{End}(\mathcal{R}_n)$, of an irreducible representation, $\mathcal{R}_n$, of $\mathrm{Spin}(2k+1)$. It contains a truncated spectrum of symmetric traceless representations of $\mathrm{SO}(2k+1)$ which form a truncated spectrum of spherical harmonics, but contains additional representations. The analysis of

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the representations in the Matrix Algebras for fuzzy spheres in diverse dimensions was given in [5]. For even spheres S^{2k} these representations span, at large n, the space of functions on a higher dimensional coset $\mathrm{SO}(2k+1)/\,\mathrm{U}(k)$, which are bundles over the spheres S^{2k}. The extra degrees of freedom in the Matrix algebra can be interpreted equivalently in terms of non-abelian fields on the spheres [6]. In the case of the fuzzy four sphere the higher dimensional geometry is $\mathrm{SO}(5)/\,\mathrm{U}(2)$ which is one realization of CP^3 as a coset.

Any discussion of field theory on the fuzzy sphere S^{2k} requires a product for the fields. The configuration space of a scalar field on the fuzzy four-sphere, is the subspace of $\mathrm{End}(\mathcal{R}_n)$ which transforms in the traceless symmetric representations. This vector subspace of $\mathrm{End}(\mathcal{R}_n)$ admits an obvious product. It is obtained by taking the Matrix product followed by a projection back to the space of Matrices transforming as symmetric traceless representations. The vector space of truncated spherical harmonics equipped with this product is denoted by $\mathcal{A}_n(S^{2k})$. The product is denoted by m_2, which can be viewed as a map from $\mathcal{A}_n(S^{2k}) \otimes \mathcal{A}_n(S^{2k})$ to $\mathcal{A}_n(S^{2k})$. It is important to distinguish $\mathrm{End}(\mathcal{R}_n)$ and $\mathcal{A}_n(S^{2k})$, which are the full Matrix algebra and the algebra obtained after projection, respectively. The product on $\mathcal{A}_n$ is non-associative and commutative but the non-associativity vanishes at large n [5, 6]. The Matrix algebra $\mathrm{End}(\mathcal{R}_n)$ contains Matrices $X_{\mu\nu}$ transforming in the adjoint of $\mathrm{SO}(2k+1)$. It also contains Matrices X_μ transforming in the vector of $\mathrm{SO}(2k+1)$. The vector of $\mathrm{SO}(2k+1)$ and the adjoint combine into the adjoint of $\mathrm{SO}(2k+2)$. The $\mathrm{SO}(2k+2)$ acts by commutators on the whole Matrix algebra.

The projection used in defining the non-associative product on $\mathcal{A}_n(S^{2k})$ commutes with $\mathrm{SO}(2k+1)$ but does not commute with $\mathrm{SO}(2k+2)$. Hence the generators of $\mathrm{SO}(2k+1)$ provide derivations on the $\mathcal{A}_n(S^{2k})$ [6]. However derivations transforming in the vector of $\mathrm{SO}(2k+1)$ would be very useful in developing gauge theory for the $\mathcal{A}_n(S^{2k})$, where we would write a covariant derivative of the form $\delta_\alpha - iA_\alpha$ and use that as a building block for the gauge theory, a technique that has found many applications in Matrix Theories [7, 8] (see for example [9]–[14]).

The classical sphere can be described in terms of an algebra generated by the coordinates z_α of the embedding R^{2k+1}, as a projection following from the constraint $\sum_{\alpha=1}^{2k+1} z_\alpha z_\alpha = 1$. When we drop the constraint of a constant radius, the algebra of functions on R^5 generated by the z_α admits derivations ∂_α which obey $\partial_\alpha z_\beta = \delta_{\alpha\beta}$. These translations can be projected to the tangent space of the sphere to give derivations on the sphere. The latter P_α can be written as $P_\alpha = \delta_\alpha - Q_\alpha$, where Q_α is a derivative transverse to the sphere, and can be characterized by its action on the cartesian coordinates.

The fuzzy sphere algebra contains analogous operators Z_α which obey a constraint $\sum_{\alpha=1}^{2k+1} Z_\alpha Z_\alpha = \frac{n+2k}{n}$ and form a finite dimensional algebra. It is natural to consider at finite n, a finite dimensional algebra generated by the Z_α obtained by dropping this quadratic constraint, but keeping the structure constants of the algebra. This can be viewed as a deformation of the algebra of polynomial functions of Z_α. We call this finite dimensional algebra $\mathcal{A}_n(R^{2k+1})$. Imposing the constraints $\sum Z_\alpha Z_\alpha = \frac{n+2k}{n}$ on this deformed polynomial algebra gives the algebra $\mathcal{A}_n(S^{2k})$. Again one can consider 'derivatives' δ_α defined such that their action on the algebra $\mathcal{A}_n(R^{2k+1})$ at large n is just the action of the generators of the translation group. In general they will not satisfy the Leibniz rule at finite n.

An approach to gauge theory on the algebra $\mathcal{A}_n(R^{2k+1})$ is to study the deformation of Leibniz Rule that is obeyed by δ_α. This will allow us to obtain the gauge transformation required of A_α such that the covariant derivative $D_\alpha = \delta_\alpha - iA_\alpha$ is indeed covariant. We can expect the non-associativity to lead to extra terms in the transformation of A_α. To get to gauge theory on $\mathcal{A}_n(S^{2k+1})$, we would define the projection operators P_α, Q_α and study their deformed Leibniz rule, and then obtain the gauge transformation rule for A_α by requiring that $P_\alpha - iA_\alpha$ is covariant. The strategy of doing non-commutative geometry on the deformed R^3 as a way of getting at the non-commutative geometry of the fuzzy S^2 is used in [15, 16].

The structure constants of the algebra $\mathcal{A}_n(S^D)$ and $\mathcal{A}_n(R^{D+1})$ show an interesting simplification at large D. We will study simpler algebras $\mathcal{B}_n(R^{2k+1})$ and $\mathcal{B}_n(S^{2k})$, which are obtained by choosing $2k+1$ generators among the $D+1$ of $\mathcal{A}_n(R^{D+1})$. By keeping k fixed and letting D be large we get a simple algebra, whose structure constants can be easily written down explicitly. This algebra is still commutative/non-associative, and allows us to explore the issues of doing gauge theory in this set-up. We explain the relation between $\mathcal{B}_n(S^{2k+1})$ and $\mathcal{A}_n(S^{2k+1})$ in section 2.

In section 3, we explore 'derivatives' on $\mathcal{B}_n(R^{2k+1})$ which obey $\delta_\alpha Z_\beta = \delta_{\alpha\beta}$. To completely define these operators we specify their action on a complete basis of the deformed polynomial algebra. The action we define reduces in the large n limit to the ordinary action of infinitesimal translations. Having defined the action of the derivatives on the space of elements in the algebra, we need to explore the interplay of the derivatives with the product. The operation δ_α obeys the Leibniz rule at large n but fails to do so at finite n. The precise deformation of the Leibniz rule is described in section 3.

In section 4 we show that by using a modified product m_2^* (which is also commutative and non-associative) the derivatives δ_α do actually obey the Leibniz rule. The modified product is related to the projected Matrix product m_2 by a 'twisting'. While our use of the term 'twist' is partly colloquial, there is some similarity to the 'Drinfeld twist' that appears in quantum groups [17, 18]. Applications of the Drinfeld twist in recent literature related to fuzzy spheres include [19]–[21].

In section 5, using the δ_α and the product m_2^* we discuss an abelian gauge theory for the finite deformed polynomial algebra $\mathcal{B}_n^*(R^{2k+1})$. The transformation law of A_α, which guarantees that D_α is covariant, now has additional terms related to the associator. A detailed description of the associator becomes necessary, some of which is given in appendix B. Appendix A is a useful prelude to the discussion of the associator. It gives the relation between the product m_2^* and the product m_2^c which is essentially the classical product on polynomials. The discussion of the gauge invariance shows a key consequence of the non-associativity, that gauge fields can pick up derivative dependences under a gauge transformation by a gauge parameter which depends only on coordinates. Hence we should enlarge our notion of gauge field to include dependences on derivatives. This naturally suggests that the gauge parameters should also be generalized to allow dependence on derivatives. We can then obtain ordinary coordinate dependent gauge fields and their gauge transformations by a partial gauge fixing. This discussion is somewhat formal since a complete discussion of gauge theory on a deformation of R^{2k+1} should include a careful discussion of the integral. We will postpone discussion of the integral for the deformed R^{2k+1} to the future. However the structure of gauge fields, gauge parameters, gauge fixing in the non-associative context uncovered here carries over to the case of the gauge theory on $\mathcal{B}_n^*(S^{2k})$ which we discuss in the next section. One new ingredient needed here is the projection of the translations of R^{2k+1} to the tangent space of the sphere. This requires introduction of new derivatives Q_α such that the projectors to the tangent space are $P_\alpha = \delta_\alpha - Q_\alpha$. The relevant properties of Q_α are described and the construction of abelian gauge theory on $\mathcal{B}_n^*(S^{2k})$ done in section 5. The discussion of the integral is much simpler in this case. Details on the deformed Leibniz rule for Q_α are given in appendix C. The final appendix D, which is not used in the paper, but is an interesting technical extension of appendix A shows how to write the classical product m_2^c in terms of m_2, the product in $\mathcal{A}_n(R^{2k+1})$

We end with conclusions and a discussion of avenues for further research.

2. The algebra $\mathcal{B}_n(R^{2k+1})$

We will define a deformed polynomial algebra $\mathcal{B}_n(R^{2k+1})$ starting from the fuzzy sphere algebras $\mathcal{A}_n(S^D)$. Consider operators in $\mathrm{End}(\mathcal{R}_n)$, where $\mathcal{R}_n$ is the irrep of $\mathrm{Spin}(D+1)$ whose highest weight is n times that of the fundamental spinor.

$$Z_{\mu_1\mu_2\cdots\mu_s} = \frac{1}{n^s} \sum_{r_1 \neq r_2 \cdots \neq r_s} \rho_{r_1}(\Gamma_{\mu_1})\rho_{r_2}(\Gamma_{\mu_2}) \cdots \rho_{r_s}(\Gamma_{\mu_s}) \,. \tag{2.1}$$

Each r_i index can run from 1 to n. The μ indices can take values from 1 to $D+1$. It is important to note that the maximum value that s can take at finite n is n.

The operators of the form (2.1) contain the symmetric traceless representation corresponding to Young Diagrams of type $(s, 0, \cdots)$ where the integers denote the first, second etc. row lengths. By contracting the indices, the operators of fixed l also contain lower representations, e.g. $(s-2, 0)$. For example

$$\begin{aligned} \sum_\mu Z_{\mu\mu} &= \frac{n(n-1)}{n^2} \\ \sum_\mu Z_{\mu\mu\alpha} &= \frac{(n-1)(n-2)}{n^2} Z_\alpha \end{aligned} \tag{2.2}$$

We now define $\mathcal{A}_n(R^{D+1})$ we keep the product structure from $\mathcal{A}_n(S^D)$ but we drop relations between Z's with contracted indices and Z's with fewer indices such as (2.2). Note that $\mathcal{A}_n(R^{D+1})$ is still finite dimensional because $s \le n$, but it a larger algebra that $\mathcal{A}_n(S^D)$. We can obtain $\mathcal{A}_n(S^D)$ from $\mathcal{A}_n(R^{D+1})$ by imposing the contraction relations. The operators of the form given in (2.1) can be obtained by multiplying Z_{μ_1} using the product in $\mathcal{A}_n(S^D)$. For example

$$Z_{\mu_1} Z_{\mu_2} = Z_{\mu_1\mu_2} + \frac{1}{n}\delta_{\mu_1\mu_2} \,. \tag{2.3}$$

More generally we have

$$Z_{\mu_1\mu_2\cdots\mu_s} = Z_{\mu_1} Z_{\mu_2} \cdots Z_{\mu_s} + \mathcal{O}\left(\frac{1}{n}\right). \tag{2.4}$$

To make clear which product we are using we write

$$m_2(Z_{\mu_1} \otimes Z_{\mu_2}) = Z_{\mu_1\mu_2} + \frac{1}{n}\delta_{\mu_1\mu_2}. \tag{2.5}$$

The m_2 denotes the product obtained by taking the Matrix product in $\mathrm{End}(\mathcal{R}_n)$ and projecting out the representations which do not transform in the symmetric traceless representations. This kind of product can be used for both $\mathcal{A}_n(R^{D+1})$ and $\mathcal{A}_n(S^D)$. Since the higher Z's are being generated from the Z_α we can view $\mathcal{A}_n(R^{D+1})$ as a deformation of the algebra of polynomials in the $D+1$ variables Z_α. As discussed previously in the context of $\mathcal{A}_n(S^D)$ [5, 6] this product is commutative and non-associative but becomes commutative and associative in the large n limit.

Now we look at a deformed R^{2k+1} subspace of the deformed R^{D+1} space. We have generators $Z_{\mu_1\mu_2\cdots\mu_s}$ where the μ indices take values from 1 to $2k+1$. This operator is symmetric under any permutation of the μ's. The largest representation of $\mathrm{SO}(2k+1)$ contained in the set of operators for fixed s is the one associated with Young Diagram of row lengths $(r_1, r_2, \cdots) = (s, 0, \cdots)$ and are symmetric traceless representations. We keep k fixed and let D be very large. We will get very simple algebras which we denote as $\mathcal{B}_n(R^{2k+1})$ and $\mathcal{B}_n(S^{2k})$. The relation between $\mathcal{B}_n(R^{2k+1})$ and $\mathcal{B}_n(S^{2k})$ is similar to that between $\mathcal{A}_n(R^{2k+1})$ and $\mathcal{A}_n(S^{2k+1})$.

Let us look at some simple examples of the product.

$$\begin{aligned} X_{\mu_1} X_{\mu_2} &= \sum_{r_1=1}^{n} \rho_{r_1}(\Gamma_{\mu_1}) \sum_{r_2=1}^{n} \rho_{r_2}(\Gamma_{\mu_2}) \\ &= \sum_{r_1 \neq r_2 = 1}^{n} \rho_{r_1}(\Gamma_{\mu_1}) \rho_{r_2}(\Gamma_{\mu_2}) + \sum_{r_1 = r_2 = 1}^{n} \rho_{r_1}(\Gamma_{\mu_1}\Gamma_{\mu_2}) \\ &= \sum_{r_1 \neq r_2 = 1}^{n} \rho_{r_1}(\Gamma_{\mu_1}) \rho_{r_2}(\Gamma_{\mu_2}) + \sum_{r_1 = r_2 = 1}^{n} \rho_{r_1}(\delta_{\mu_1\mu_2}) = X_{\mu_1\mu_2} + n\delta_{\mu_1\mu_2}. \end{aligned} \tag{2.6}$$

In the first line we have written the product of two X operators which are both Γ matrices acting on each of n factors of a symmetrized tensor product of spinors [2]. In the second line we have separated the double sum into terms where $r_1 = r_2$ and where they are different. In other words we are looking separately at terms where the two Γ's act on the same tensor factor and when they act on different tensor factors. Where $r_1 \neq r_2$ the expression is symmetric under exchange of μ_1 and μ_2. When $r_1 = r_2$, there is a symmetric part and an antisymmetric part. The antisymmetric part is projected out when we want to define the product which closes on the fuzzy spherical harmonics [5]. This is why the third line only keeps $\delta_{\mu_1\mu_2}$. Expressing the product in terms of the normalized Z_μ we have

$$Z_{\mu_1} Z_{\mu_2} = Z_{\mu_1\mu_2} + \frac{1}{n}\delta_{\mu_1\mu_2}. \tag{2.7}$$

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By similar means we compute

$$Z_{\mu_1\mu_2}Z_{\mu_3} = Z_{\mu_1\mu_2\mu_3} + \frac{(n-1)}{n^2}(\delta_{\mu_1\mu_3}Z_{\mu_2} + \delta_{\mu_2\mu_3}Z_{\mu_1})\,. \tag{2.8}$$

The l.h.s. contains sums of the form

$$\sum_{r_1\neq r_2}\sum_{r_3} = \sum_{r_1\neq r_2\neq r_3} + \sum_{r_3=r_1\neq r_2} + \sum_{r_1\neq r_2=r_3}\,. \tag{2.9}$$

The three types of sums in (2.9) lead, respectively, to the first, second and third terms on the r.h.s. of (2.8). The factor of $n-1$ in the second term, for example, comes from the fact that the $r_1 = r_2$ sum runs from 1 to n avoiding the single value r_3. The $1/n^2$ comes from normalizations. The relations (2.7) and (2.8) hold both in $\mathcal{B}_n(R^{2k+1})$ and $\mathcal{A}_n(R^{2k+1})$.

Using (2.7) and (2.8) it is easy to see the non-associativity. Indeed

$$\begin{aligned}
(Z_{\mu_1}Z_{\mu_2})Z_{\mu_3} &= Z_{\mu_1\mu_2\mu_3} + \frac{(n-1)}{n^2}(\delta_{\mu_1\mu_3}Z_{\mu_2} + \delta_{\mu_2\mu_3}Z_{\mu_1}) + \frac{1}{n}\delta_{\mu_1\mu_2}Z_{\mu_3}\\
Z_{\mu_1}(Z_{\mu_2}Z_{\mu_3}) &= Z_{\mu_1\mu_2\mu_3} + \frac{(n-1)}{n^2}(\delta_{\mu_1\mu_3}Z_{\mu_2} + \delta_{\mu_1\mu_2}Z_{\mu_3}) + \frac{1}{n}\delta_{\mu_2\mu_3}Z_{\mu_1}\\
(Z_{\mu_1}Z_{\mu_2})Z_{\mu_3} - Z_{\mu_1}(Z_{\mu_2}Z_{\mu_3}) &= \frac{1}{n^2}(\delta_{\mu_1\mu_2}Z_{\mu_3} - \delta_{\mu_2\mu_3}Z_{\mu_1})\,.
\end{aligned} \tag{2.10}$$

We now explain the simplification that arises in $\mathcal{B}_n(R^{2k+1})$ as opposed to $\mathcal{A}_n(R^{2k+1})$. Consider a product

$$Z_{\mu_1\mu_2}Z_{\mu_3\mu_4} = \frac{1}{n^4}\sum_{r_1\neq r_2}\rho_{r_1}(\Gamma_{\mu_1})\rho_{r_2}(\Gamma_{\mu_2})\cdot\sum_{r_3\neq r_4}\rho_{r_3}(\Gamma_{\mu_3})\rho_{r_4}(\Gamma_{\mu_4})\,. \tag{2.11}$$

In $\mathcal{A}_n(R^{2k+1})$ or $\mathcal{A}_n(S^{2k})$, we will get terms of the form $Z_{\mu_1\mu_2\mu_3\mu_4}$ where $r_1 \neq r_2 \neq r_3 \neq r_4$. In addition there will be terms of the form $\delta_{\mu_1\mu_3}Z_{\mu_2\mu_4}$ from terms where we have $r_1 = r_3$ and $r_2 \neq r_4$. If $r_1 = r_3 \neq r_2 \neq r_4$, the antisymmetric part of $\Gamma_{\mu_1}\Gamma_{\mu_3}$ appears in the Matrix product of $Z_{\mu_1\mu_2}$ with $Z_{\mu_3\mu_4}$ but transforms as the representation $\vec{r} = (2,1,0,\ldots)$. Hence these are projected out. However when we consider terms coming from $r_1 = r_3$ and $r_2 = r_4$ and take the operators of the form $\rho_{r_1}([\Gamma_{\mu_1},\Gamma_{\mu_3}])\rho_{r_2}([\Gamma_{\mu_2},\Gamma_{\mu_4}])$, these will include representations of type $(2,2,0,\cdots)$ which are projected out, in addition to those of type $(2,0,0\ldots)$. The latter come from the trace parts $\delta_{\mu_1\mu_2}$. They have coefficients that go as $1/D$ and hence disappear in the large D limit. We conjecture that all terms of this sort, which transform as symmetric traceless reps, but come from antisymmetric parts of products in coincident r factors, are subleading in a $1/D$ expansion. The simple product on the $Z_{\mu_1\mu_2\cdots}$, obtained by dropping all the antisymmetric parts in the coincident r factors (or according to the conjecture, by going to the large D fixed k limit) gives an algebra we denote as $\mathcal{B}_n(R^{2k+1})$. We will not try to give a proof of the conjecture that $\mathcal{B}_n(R^{2k+1})$ arises indeed in the large D limit $\mathcal{A}_n(R^{D+1})$ as outlined above. We may also just study $\mathcal{B}_n(R^{2k+1})$ as an algebra that has many similarities with $\mathcal{A}_n(R^{2k+1})$ but has simpler structure constants, while sharing the key features of being a commutative non-associative algebra which approaches the ordinary polynomial algebra of functions on R^{2k+1} in the large n limit.

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We now give an explicit description of the product on elements $Z_{\mu_1\cdots\mu_s}$ in the algebra $\mathcal{B}_n(R^{2k+1})$. It will be convenient to set up some definitions as a prelude to a general formula. For a set of integers S we will denote by $Z_{\mu(S)}$ an operator of the form (2.1) with the labels on the μ's taking values in the set S. For example $Z_{\mu_1\cdots\mu_s}$ is of the form $Z_{\mu(S)}$ with the set S being the set of integers ranging from 1 to s. Instead of writing $Z_{\mu_1\mu_4\mu_5}$ we can write $Z_{\mu(S)}$ with S being the set of integers $\{1,4,5\}$.

Given two sets of positive integers T_1 and T_2 where T_1 and T_2 have the same number of elements, which we denote as $|T_1| = |T_2| \equiv t$, we define

$$\tilde{\delta}(\mu(T_1)\mu(T_2)) = \sum_{\sigma} \delta(\mu_{i_1}\mu_{j_{\sigma(1)}}) \cdots \delta(\mu_{i_t}\mu_{j_{\sigma(t)}}a) \,. \tag{2.12}$$

The δ's on the r.h.s. are ordinary Kronecker deltas. The i's are the integers of the set T_1 with a fixed ordering, which can be taken as $i_1 < i_2 < \cdots < i_t$. The j's are the integers of the set T_2 with a similar ordering. The σ in the sum runs over all the permutations in the group of permutations of t elements. The $\tilde{\delta}$ is therefore a sum over all the $t!$ ways of pairing the integers in the set T_1 with those in the set T_2.

With this notation the general formula for the product can be expressed as

$$\begin{aligned} Z_{\mu(S)}Z_{\mu(T)} = &\sum_{|T_1|}\sum_{|T_2|} \delta(|T_1|,|T_2|) \sum_{T_1\subset S}\sum_{T_2\subset T} \times \\ &\times \frac{1}{n^{2|T_1|}} \frac{(n-|S|-|T|+2|T_1|)!}{(n-|S|-|T|+|T_1|)!} \tilde{\delta}(\mu(T_1),\mu(T_2)) Z_{\mu(S\cup T\setminus T_1\cup T_2)} \,. \end{aligned} \tag{2.13}$$

We have chosen S and T to be the sets describing two elements of $\mathcal{B}_n$ of the form (2.1). There is a sum over positive integers $|T_1|$ and $|T_2|$ which are the cardinalities of subsets T_1 and T_2 of S and T respectively. Given the restriction $|T_1| = |T_2|$, the sum over $|T_1|$ extends from 0 to $\min(|S|,|T|)$. The factor expressed in terms of factorials comes from the $|T_1|$ summations which can be done after replacing quadratic expressions in Γ's by δ's. The formula expresses the fact that the different terms in the product are obtained by summing over different ways of picking subsets T_1 and T_2 of S and T which contain the elements that lead to δ's. The set of remaining elements $S \cup T \setminus T_1 \cup T_2$ gives an operator of the form (2.1). It is instructive to check that (2.7) and (2.8) are special cases of (2.13). Note that we should set to zero all $Z_{\mu(S)}$ where $|S| > n$. For example this leads to a restriction on the sum over $|T_1|$ requiring it to start at $max(0, |S| + |T| - n)$. This is not an issue if we are doing the $1/n$ expansion.

3. Deformed derivations on $\mathcal{B}_n(R^{2k+1})$

We now define operators δ_α, which are derivations in the classical limit, by giving their action on the above basis.

$$\delta_\alpha Z_{\mu_1\mu_2\cdots\mu_s} = \sum_{i=1}^{s} \delta_{\alpha\mu_i} Z_{\mu(1\ldots s\setminus i)} \,. \tag{3.1}$$

At finite n these are not derivations. Rather they are deformed derivations, which can described in terms of a co-product, a structure which comes up for example in describing the action of quantum enveloping algebras on tensor products of representations or on products of elements of a q-deformed space of functions (see for example [22, 23])

$$\begin{aligned}\Delta(\delta_\alpha) &= (\delta_\alpha \otimes 1 + 1 \otimes \delta_\alpha) \sum_{k=0}^{\infty} \frac{(-1)^k}{n^{2k}} \delta_{\alpha_1}\delta_{\alpha_2}\cdots\delta_{\alpha_k} \otimes \delta_{\alpha_1}\delta_{\alpha_2}\cdots\delta_{\alpha_k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{n^{2k}} \delta_\alpha\delta_{\alpha_1}\delta_{\alpha_2}\cdots\delta_{\alpha_k} \otimes \delta_{\alpha_1}\delta_{\alpha_2}\cdots\delta_{\alpha_k} + \\ &\quad + \sum_{k=0}^{\infty} \frac{(-1)^k}{n^{2k}} \delta_{\alpha_1}\delta_{\alpha_2}\cdots\delta_{\alpha_k} \otimes \delta_\alpha\delta_{\alpha_1}\delta_{\alpha_2}\cdots\delta_{\alpha_k}\,. \end{aligned} \tag{3.2}$$

The leading $k = 0$ terms just lead to the ordinary Leibniz rule, and the corrections are subleading as $n \to \infty$. These formulae show that co-product is a useful structure for describing the deformation of Leibniz rule. Another possibility one might consider is to see if adding $1/n$ corrections of the type $\delta_\alpha(1 + \frac{1}{n^2}\delta_\beta\delta_\beta)$ gives a derivative which obeys the exact Leibniz rule. This latter possibility does not seem to work.

The co-product (3.2) has the property that

$$\delta_\alpha \cdot m_2 = m_2 \cdot \Delta(\delta_\alpha)\,. \tag{3.3}$$

Another way of expressing this is that

$$\begin{aligned}\delta_\alpha(A\cdot B) &= m_2 \cdot \Delta(\delta_\alpha) A \otimes B \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{n^{2k}} \delta_\alpha\delta_{\alpha_1}\delta_{\alpha_2}\cdots\delta_{\alpha_k} A \cdot \delta_{\alpha_1}\delta_{\alpha_2}\cdots\delta_{\alpha_k} B + \\ &\quad + \sum_{k=0}^{\infty} \frac{(-1)^k}{n^{2k}} \delta_{\alpha_1}\delta_{\alpha_2}\cdots\delta_{\alpha_k} A \cdot \delta_\alpha\delta_{\alpha_1}\delta_{\alpha_2}\cdots\delta_{\alpha_k} B\,. \end{aligned} \tag{3.4}$$

The proof is obtained by evaluating l.h.s. and r.h.s. on arbitrary pair of elements in the algebra.

It is useful to summarize some properties of δ_α which act as deformed derivations on the *deformed polynomial algebra* $\mathcal{A}_n(R^{2k+1})$ They approach ∂_α in the large n limit. At finite n they obey $[\delta_\alpha, \delta_\beta] = 0$ At finite n they transform as a vector under the SO$(2k+1)$ Lie algebra of derivations that acts on $\mathcal{B}_n$, i.e we have

$$[L_{\mu\nu}, \delta_\alpha] = \delta_{\nu\alpha}\delta_\mu - \delta_{\mu\alpha}\delta_\nu\,. \tag{3.5}$$

These properties can be checked by acting with the l.h.s. and r.h.s. on a general element of the algebra.

3.1 A useful identity on binomial coefficients

In proving that the above is the right co-product we find the following to be a useful identity.

$$\sum_{k=0}^{A} (-1)^k \frac{(N+1+2A)!}{(A-k)!(N+1+A+k)!} = \frac{(N+2A)!}{A!(N+A)!}\,. \tag{3.6}$$

This is a special case of the following equation from [24]

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\frac{(r+b-1)!}{(r+a-1)!}=\frac{(n+a-b-1)!(b-1)!}{(n+a-1)!(a-b-1)!}\,. \tag{3.7}$$

Choose $n = A$, $b = 1$ and $a = N + A + 2$ in (3.7) to get precisely the desired equation (3.6).

The proof of the deformed derivation property (3.4) proceeds by writing out both sides of the equation for a general pair of elements A, B of the form $Z_{\mu(S)}$ and $Z_{\mu(T)}$.

For the l.h.s. where we multiply first and take derivatives after, that is we compute $\delta_\alpha .m_2 \cdot A\otimes B$ we have

$$\begin{aligned}\delta_\alpha\cdot m_2\cdot Z_{\mu(S)}\otimes Z_{\mu(T)} = &\sum_{|S_1|}\sum_{|S_2|}\sum_{S_1\subset S}\quad\sum_{S_2\subset T}\quad\sum_{i\subset\{S\cup T\}\backslash\{S_1\cup S_2\}}\delta(|S_1|,|S_2|)\times\\ &\times\tilde{\delta}(\mu(S_1),\mu(S_2))\delta_{\alpha\mu_i}\frac{((n-|S|-|T|+2|S_1|)!}{(n-|S|-|T|+|S_1|)!}\times\\ &\times Z_{\mu(\{S\cup T\}\backslash\{S_1\cup S_2\cup i\})}\,.\end{aligned} \tag{3.8}$$

For the r.h.s. of (3.4) we have

$$\begin{aligned}&m_2\cdot\Delta(\delta_\alpha)\cdot(Z_{\mu(S)}\otimes Z_{\mu(T)}) =\\ &=\sum_{|T_1|}\sum_{|T_2|}\sum_{|T_3|}\sum_{|T_4|}\ \delta\ (|T_1|\ ,\ |T_2|)\ \delta\ (|T_3|\ ,\ |T_4|)\times\\ &\quad\times\sum_{T_1\subset S,T_2\subset T}\quad\sum_{T_3\subset S\backslash T_1,T_4\subset T\backslash T_2}\quad\sum_{i\subset\{S\cup T\}\backslash\{T_1\cup T_2\cup T_3\cup T_4\ \}}\times\\ &\quad\times(-1)^{|T_1|}|T_1|!\frac{(n-|S|-|T|+2|T_1|+2|T_3|+1)!}{(n-|S|-|T|+2|T_1|+|T_3|+1)!}\times\\ &\quad\times\delta_{\alpha\mu_i}\tilde{\delta}(\mu(T_1),\mu(T_2))\tilde{\delta}(\mu(T_3),\mu(T_4))Z_{\mu(\{S\cup T\}\backslash\{T_1\cup T_2\cup T_3\cup T_4\cup i\})}\,.\end{aligned} \tag{3.9}$$

This follows from the the definition of $\Delta(\delta_\alpha)$ given in (3.2), the action of δ_α described in (3.1), and the general structure of the product described in (2.13). The μ indices associated to the pair of subsets (T_1, T_2) are contracted with Kronecker deltas coming from the tensor product $\delta_{\alpha_1}\delta_{\alpha_2}\cdots\delta_{\alpha_{|T_1|}}\otimes\delta_{\alpha_1}\delta_{\alpha_2}\cdots\delta_{\alpha_{|T_1|}}$ appearing in $\Delta(\delta_\alpha)$. The $|T_1|!$ arises because the same set of μ contractions can come from different ways of applying the δ_α's in $\Delta(\delta_\alpha)$. The contractions involving the sets T_3 and T_4 of indices come from the multiplication m_2. We will manipulate (3.9) to show equality with (3.8).

We write $S_1 = T_1 \cup T_3 \subset S$ and $S_2 = T_2 \cup T_4 \subset T$. We observe that the Z's in (3.9) only depend on these subsets. We have of course the relations $|S_1| = |T_1| + |T_3|$ and $|S_2| = |T_2| + |T_4|$ and $|S_1| = |S_2|$. For fixed subsets S_1, S_2 we observe the identity :

$$\begin{aligned}&\sum_{T_1\subset S_1\ ,\ T_2\subset S_2}\quad\sum_{T_3\subset S_1\backslash T_1\ ,\ T_4\subset S_2\backslash T_2}\tilde{\delta}(\mu(T_1),\mu(T_2))\tilde{\delta}(\mu(T_3),\mu(T_4)) =\\ &\qquad=\frac{(|T_1|+|T_3|)!}{|T_1|!|T_3|!}\tilde{\delta}(\mu(S_1),\mu(S_2))\,.\end{aligned} \tag{3.10}$$

For fixed sets S_1, S_2, the $\tilde{\delta}(\mu(S_1),\mu(S_2))$ is a sum of $|S_1|!$ terms. The sums on the l.h.s. contain $\binom{|S_1|}{|T_1|}\binom{|S_2|}{|T_2|} = \binom{|S_1|}{|T_1|}^2$ choices of T_1, T_2, and the two $\tilde{\delta}$'s on the l.h.s. contain a sum of $|T_1|!|T_2|! = (|T_1|!)^2$ terms, each being a product of Kronecker deltas. The combinatoric factor in (3.10) is the ratio $\frac{\binom{|S_1|}{|T_1|}\binom{|S_2|}{|T_2|}|T_1|!|T_3|!}{|S_1|!}$. Given that the summands can be written in terms of S_1 and S_2, the summations over the cardinalities of the T_i subsets can be rewritten using

$$\sum_{|T_1|}\sum_{|T_2|}\sum_{|T_3|}\sum_{|T_4|}\delta(|T_1|,|T_2|)\delta(|T_3|,|T_4|) = \sum_{|S_1|}\sum_{|S_2|}\delta(|S_1|,|S_2|)\,. \tag{3.11}$$

Using (3.10) and (3.11) we can rewrite (3.9) as

$$\begin{aligned}
&\sum_{|S_1|}\sum_{|S_2|}\delta(|S_1|,|S_2|)\sum_{|T_1|=0}^{|S_1|}(-1)^{|T_1|}|T_1|!\frac{(|T_1|+|T_3|)!}{|T_1|!|T_3|!}\times\\
&\times\sum_{S_1\subset S}\sum_{S_2\subset T}\tilde{\delta}(\mu(S_1),\mu(S_2))\sum_{i\subset\{S\cup T\}\setminus\{S_1\cup S_2\}}\delta_{\alpha\mu_i}\times\\
&\times\frac{(N+2|T_1|+2|T_3|+1)!}{(N+2|T_1|+|T_3|+1)!}Z_{\mu(\{S\cup T\}\setminus\{S_1\cup S_2\cup i\})}\,.
\end{aligned}\tag{3.12}$$

To simplify the combinatoric factors we have defined $N \equiv n-|S|-|T|$. We can further simplify as follows

$$\begin{aligned}
&\sum_{|S_1|}\left(\sum_{|T_1|=0}^{|S_1|}(-1)^{|T_1|}\frac{|S_1|!}{(|S_1|-|T_1|)!}\frac{(N+2|S_1|+1)!}{(N+|S_1|+|T_1|+1)!}\right)\times\\
&\times\sum_{S_1\subset S}\sum_{S_2\subset T}\sum_{i\subset\{S\cup T\}\setminus\{S_1\cup S_2\}}\tilde{\delta}(\mu(S_1),\mu(S_2))\delta_{\alpha\mu_i}Z_{\mu(\{S\cup T\}\setminus\{S_1\cup S_2\cup i\})} =\\
&\quad=\sum_{|S_1|}\frac{(N+2|S_1|)!}{(N+|S_1|)!}\sum_{S_1\subset S}\sum_{S_2\subset T}\sum_{i\subset\{S\cup T\}\setminus\{S_1\cup S_2\}}\times\\
&\qquad\times\tilde{\delta}(\mu(S_1),\mu(S_2))\delta_{\alpha\mu_i}Z_{\mu(\{S\cup T\}\setminus\{S_1\cup S_2\cup i\})}\,.
\end{aligned}\tag{3.13}$$

In the final equality we have used (3.6) by setting $A = |S_1|$ and $k = |T_1|$. This establishes the equality of (3.8) and (3.9).

4. Derivations for the twisted algebra $\mathcal{B}^*_n(R^{2k+1})$

We defined above certain deformed derivations. This is reminiscent of quantum groups where the quantum group generators act on tensor products via a deformation of the usual co-product. In this context, classical and quantum co-products can be related by a Drinfeld twist [18], in which an element F living in $U \otimes U$ plays a role, where U is the quantum enveloping algebra. This suggests that we could twist the product m_2 of the algebra $\mathcal{B}_n(R^{2k+1})$ to get a new product m^*_2 which defines a new algebra $\mathcal{B}^*_n(R^{2k+1})$. These two algebras share the same underlying vector space. We want to define a star product m^*_2 which is a twisting of the product m_2, for which the δ_α are really derivations rather than deformed derivations. It turns out that this becomes possible after we use in addition to the δ_α a degree operator D. For symmetric elements of the form $Z_{\mu(S)}$ the degree operator is defined to have eigenvalue $|S|$. In the following we will describe a twist which uses D, δ_α and leads to a product m_2^* for which the δ_α are really derivations. It will be interesting to see if the formulae given here are related to Drinfeld twists in a precise sense. The physical idea is that field theory actions can be written equivalently either using m_2 or m_2^*, somewhat along the lines of the Seiberg-Witten map [25].

4.1 Formula for the star product

Let us write m_2^* for the star product. It is useful to view m_2^* as a map from $\mathcal{A}_n \otimes \mathcal{A}_n$ to $\mathcal{A}_n$. With respect to the new product m_2^*, δ_α obey the Leibniz rule at finite n.

$$\delta_\alpha \cdot m_2^* = m_2^* \cdot (\delta_\alpha \otimes 1 + 1 \otimes \delta_\alpha) \,. \tag{4.1}$$

Equivalently for two arbitrary elements A and B of the algebra, we have, after writing $m_2^*(A \otimes B) = A * B$,

$$\delta_\alpha(A * B) = (\delta_\alpha A) * B + A * (\delta_\alpha B) \,. \tag{4.2}$$

We look for a formula for m_2^* as an expansion in terms of m_2, the derivatives δ_α, and using a function of the degree operator to be determined. The ansatz takes the form

$$m_2^* = \sum_{l=0}^{\infty} \frac{1}{n^{2l}} h_l(D) \cdot m_2 \cdot \delta_{\alpha_1}\delta_{\alpha_2}\cdots\delta_{\alpha_l} \otimes \delta_{\alpha_1}\delta_{\alpha_2}\cdots\delta_{\alpha_l} \,. \tag{4.3}$$

The function $h_l(D)$ is determined by requiring that (4.2) hold for arbitrary A and B. It turns out, as we show later, that

$$h_l(D) = \binom{D}{l} \,. \tag{4.4}$$

For finite n the degrees of the operators do not exceed n. We can hence restrict the range of summation in (4.3) from 0 to n.

A useful identity in proving that with the choice (4.4) the product in (4.3) satisfies the Leibniz rule is

$$\sum_{l=0}^{p_1} h_l(s-1)\binom{n-s+1}{p_1-l} = \sum_{l=0}^{p_1} h_l(s)\binom{n-s}{p_1-l} \,. \tag{4.5}$$

This follows from the combinatoric identity (see for example [26]) which can be written in the suggestive form

$$\sum_{l=0}^{p_1} \binom{N}{k}\binom{M}{p_1-k} = \binom{N+M}{p_1} \,. \tag{4.6}$$

Substituting N with $s-1$ and M with $n-s+1$ we have

$$\sum_{l=0}^{p_1} \binom{s-1}{l}\binom{n-s+1}{p_1-l} = \binom{n}{p_1} \,. \tag{4.7}$$

Substituting N with s and M with $n-s$ we have

$$\sum_{l=0}^{p_1} \binom{s}{l} \binom{n-s}{p_1-l} = \binom{n}{p_1} . \tag{4.8}$$

Hence the identity (4.5) required of f is indeed satisfied by the choice (4.4).

4.2 Proof of the derivation property for the twisted product

We compute $\delta_\alpha \cdot m_2^* \cdot (Z_{\mu(S)} \otimes Z_{\mu(T)})$ using the definitions (4.3), (3.1) and the general form of the m_2 product in (2.13),to obtain

$$\begin{aligned}
&\sum_{l=0}^{\infty} \sum_{|T_1|} \sum_{|T_2|} \delta(|T_1|, l) \delta(|T_2|, l) \sum_{|T_3|} \sum_{T_4|} \delta(|T_3|, |T_4|) \times \\
&\times \sum_{T_1 \subset S} \sum_{T_2 \subset T} \sum_{T_3 \subset S \backslash T_1} \sum_{T_4 \subset T \backslash T_2} \sum_{i \in \{S \cup T\} \backslash \{T_1 \cup T_2 \cup T_3 \cup T_4\}} \frac{l!}{n^{2l}} h_l \times \\
&\times (|S| + |T| - 2l - 2|T_3|) \frac{1}{n^{2|T_3|}} \frac{(n - |S| - |T| + 2l + 2|T_3|)!}{(n - |S| - |T| + 2l + |T_3|)!} \times \\
&\times \tilde{\delta}(\mu(T_1), \mu(T_2)) \; \tilde{\delta}(\mu(T_3), \mu(T_4)) \delta_{\mu \alpha_i} Z_{\mu(\{S \cup T\} \backslash \{T_1 \cup T_2 \cup T_3 \cup T_4 \cup i\})} \, .
\end{aligned} \tag{4.9}$$

Similarly we compute $m_2^* \cdot (\delta_\alpha \otimes 1 + 1 \otimes \delta_\alpha) \cdot (Z_{\mu(S)} \otimes Z_{\mu(T)})$ to obtain

$$\begin{aligned}
&\sum_{l=0}^{\infty} \sum_{|T_1|} \sum_{|T_2|} \delta(|T_1|, l) \delta(|T_2|, l) \sum_{|T_3|} \sum_{T_4|} \delta(|T_3|, |T_4|) \times \\
&\times \Bigg(\sum_{j \in S} \sum_{T_1 \subset S \backslash j} \sum_{T_2 \subset T} \sum_{T_3 \subset S \backslash j \cup T_1} \sum_{T_4 \subset T \backslash T_2} + \\
&\qquad + \sum_{j \in T} \sum_{T_1 \subset S} \sum_{T_2 \subset T \backslash j} \sum_{T_3 \subset S \backslash T_1} \sum_{T_4 \subset T \backslash j \cup T_2} \Bigg) \times \\
&\times \delta_{\alpha \mu_j} \tilde{\delta}(\mu(T_1), \mu(T_2)) \; \tilde{\delta}(\mu(T_3), \mu(T_4)) Z_{\mu(\{S \cup T\} \backslash \{j \cup T_1 \cup T_2 \cup T_3 \cup T_4\})} \times \\
&\times \frac{l!}{n^{2l}} \frac{1}{n^{2|T_3|}} \frac{(n - |S| - |T| + 2l + 2|T_3| + 1)!}{(n - |S| - |T| + 2l + |T_3| + 1)!} h_l(|S| + |T| - 2|T_1| - 2|T_3| - 1) \, .
\end{aligned} \tag{4.10}$$

At finite n the number of linearly independent operators Z we need to consider are bounded by n, i.e. $|S| \leq n$ and $|T| \leq n$. This means that the sum over l is also a finite sum.

Observe in (4.10) that the Z only depends on the unions $T_1 \cup T_3$ and $T_2 \cup T_4$. We denote $S_1 \equiv T_1 \cup T_3$ and $S_2 = T_2 \cup T_4$. It follows that the cardinalities are related $|S_1| = |T_1| + |T_3|$ and $|S_2| = |T_2| + |T_4|$. The sums over subsets in (4.10) can be rearranged as

$$\sum_{|S_1|} \sum_{|S_2|} \delta(|S_1|, |S_2|) \sum_{S_1 \subset S} \sum_{S_2 \subset T} \sum_{j \in \{S \cup T\} \backslash \{S_1 \cup S_2\}} \, . \tag{4.11}$$

Hence we re-express (4.10) as

$$\sum_{|S_1|}\sum_{|S_2|}\delta(|S_1|,|S_2|)\sum_{S_1\subset S}\sum_{S_2\subset T}\sum_{j\in\{S\cup T\}\setminus\{S_1\cup S_2\}}\sum_{l=0}^{|S_1|}\tilde{\delta}(\mu(S_1),\mu(S_2))\times$$
$$\times\,\delta_{\alpha\mu_j}Z_{\mu(\{S\cup T\}\setminus\{S_1\cup S_2\cup j\})}\frac{(n-|S|-|T|+2l+2|T_3|+1)!}{(n-|S|-|T|+2l+|T_3|+1)!}l!\frac{1}{n^{2|T_3|+2l}}\frac{|S_1|!}{l!(|S_1|-l)!}\,. \quad (4.12)$$

The binomial factor $\frac{|S_1|!}{l!(|S_1|-l)!}$ appears in using the conversion (3.10) of a sum of products of $\tilde{\delta}$'s to a single $\tilde{\delta}$. Simplifying (4.11)

$$\sum_{|S_1|}\sum_{|S_2|}\delta(|S_1|,|S_2|)\sum_{S_1\subset S}\sum_{S_2\subset T}\sum_{j\in\{S\cup T\}\setminus\{S_1\cup S_2\}}\frac{1}{n^{2|S_1|}}\times$$
$$\times\,\tilde{\delta}(\mu(S_1),\mu(S_2))\delta_{\alpha\mu_j}Z_{\mu(\{S\cup T\}\setminus\{S_1\cup S_2\cup j\})}\times$$
$$\times\left(\sum_{l=0}^{|S|_1}\frac{(n-|S|-|T|+2|S_1|+1)!}{(n-|S|-|T|+|S_1|+l+1)!}h_l(|S|+|T|-2|S_1|-1)\frac{|S_1|!}{(|S_1|-l)!}\right). \quad (4.13)$$

Similar manipulations can be done starting from (4.9) to collect the products of $\tilde{\delta}$'s into a single $\tilde{\delta}$. We end up with

$$\sum_{|S_1|}\sum_{|S_2|}\delta(|S_1|,|S_2|)\sum_{S_1\subset S}\sum_{S_2\subset T}\sum_{j\in\{S\cup T\}\setminus\{S_1\cup S_2\}}\frac{1}{n^{2|S_1|}}\times$$
$$\times\tilde{\delta}(\mu(S_1),\mu(S_2))\delta_{\alpha\mu_j}Z_{\mu(\{S\cup T\}\setminus\{S_1\cup S_2\cup j\})}\times$$
$$\times\left(\sum_{l=0}^{|S|_1}\frac{(n-|S|-|T|+2|S_1|)!}{(n-|S|-|T|+|S_1|+l)!}h_l(|S|+|T|-2|S_1|)\frac{|S_1|!}{(|S_1|-l)!}\right). \quad (4.14)$$

This expression is the same as (4.13) except in the sum appearing in the last line. Since (4.13) has the derivative acting before the m_2^*, the $h_l(D)$ in the m_2^* has an argument which is the degree $|S|+|T|-2|S_1|-1$. This is because $Z_{\mu(S\cup T)}$ has degree $|S|+|T|$ and the degree gets reduced by $2|S_1|=2|T_1|+2|T_3|$ due to the $|T_1|$ contractions from the derivatives in m_2^* and the $|T_3|$ contractions from the product m_2 in the expression for m_2^*. Finally the reduction by 1 is due to the δ_α which has already acted before the m_2^* and hence before $h_l(D)$. In (4.14) the argument of h_l is larger by 1 because δ_α acts before m_2^*. The ratio of factorials also has a relative shift of 1 because they arise from m_2 according to (2.13) and the m_2 acts before the δ_α in one case and after it in the other. The validity of (4.1) or equivalently the equality of (4.9) and (4.10) will follow from the equality of the sums

$$\sum_{l=0}^{|S_1|}\frac{(n-|S|-|T|+2|S_1|)!}{(n-|S|-|T|+|S_1|+l)!}h_l(|S|+|T|-2|S_1|)\frac{|S_1|!}{(|S_1|-l)!}=$$
$$=\sum_{l=0}^{|S|_1}\frac{(n-|S|-|T|+2|S_1|+1)!}{(n-|S|-|T|+|S_1|+l+1)!}h_l(|S|+|T|-2|S_1|-1)\frac{|S_1|!}{(|S_1|-l)!}\,. \quad (4.15)$$

Substituting $h_l(D) = \binom{D}{l}$ and defining $u = |S| + |T| - 2|S_1|$ to simplify formulae, the l.h.s. of (4.15) becomes

$$\sum_{l=0}^{|S|_1} \frac{(n-u)!}{(n-u-|S_1|+l)!} \frac{u!}{l!(u-l)!} \frac{|S_1|!}{(|S_1|-l)!} = |S_1|! \sum_{l=0}^{|S_1|} \binom{n-u}{|S_1|-l} \binom{u}{l} = \frac{n!}{(n-|S_1|)!} \, . \tag{4.16}$$

In the last step we used (4.6) (see for example [26]) with $N = u$ and $M = (n-u)$. For the r.h.s. of (4.15) we get

$$\sum_{l=0}^{|S_1|} \frac{(n-u+1)!}{(n-u-|S_1|+l+1)!} \frac{(u-1)!}{l!(u-l-1)!} \frac{|S_1|!}{(|S_1|-l)!} = |S_1|! \sum_{l=0}^{|S_1|} \binom{n-u+1}{|S_1|-l} \binom{u-1}{l} = \frac{n!}{(n-|S_1|)!} \, . \tag{4.17}$$

Here we have used (4.6) with $N = u-1$ and $M = n-u+1$. This establishes the equation (4.1), and also shows that (4.9) and (4.10) can be simplified to

$$\sum_{|S_1|} \sum_{|S_2|} \delta(|S_1|,|S_2|) \sum_{S_1 \subset S} \sum_{S_2 \subset T} \sum_{j \in \{S \cup T\} \backslash \{S_1 \cup S_2\}} \frac{1}{n^{2|S_1|}} \times \times \tilde{\delta}(\mu(S_1),\mu(S_2)) \delta_{\alpha\mu_j} Z_{\mu(\{S\cup T\} \backslash \{S_1 \cup S_2 \cup j\})} \frac{n!}{(n-|S_1|)!} \, . \tag{4.18}$$

It is also useful to note, as a corollary, that

$$Z_{\mu(S)} * Z_{\mu(T)} = \sum_{|S_1|} \sum_{|S_2|} \delta(|S_1|,|S_2|) \sum_{S_1 \subset S} \sum_{S_2 \subset T} \frac{n!}{(n-|S_1|)!} \frac{1}{n^{2|S_1|}} \times \times \tilde{\delta}(\mu(S_1),\mu(S_2)) Z_{\mu(S\cup T \backslash S_1 \cup S_2)} \, . \tag{4.19}$$

5. Towards abelian gauge theory for $\mathcal{B}_n^*(R^{2k+1})$

5.1 Gauge transformations and non-associativity

The construction of gauge theory starts with the definition of covariant derivatives $D_\alpha = \delta_\alpha - iA_\alpha$. These are used to define covariant field strengths which lead to invariant actions. Consider some field Φ in the fundamental of the gauge group. Under a gauge transformation by a gauge parameter ϵ the variation is given, as usual, by

$$\delta\Phi = i\epsilon * \Phi \, . \tag{5.1}$$

The desired covariance property of the proposed covariant derivative is

$$\delta(D_\mu\Phi) = \epsilon * (D_\mu\Phi) \, . \tag{5.2}$$

We look for the transformation δA_μ which will guarantee this covariance. Expanding $\delta(D_\mu \Phi)$ we get

$$\begin{aligned}
\delta(D_\mu \Phi) &= \delta((\delta_\mu - iA_\mu) * \Phi) \\
&= i\delta_\mu(\epsilon * \Phi) - i(\delta A_\mu) * \Phi + A_\mu * (\epsilon * \Phi) \\
&= i(\delta_\mu \epsilon) * \Phi + i\epsilon * (\delta_\mu \Phi) - i(\delta A_\mu) * \Phi + A_\mu * (\epsilon * \Phi)\,. \qquad (5.3)
\end{aligned}$$

In the last line we used the fact the derivatives δ_μ obey the correct Leibniz rule with respect to the star product (4.1). This manipulation would be more complicated if we used the product m_2, as indicated by (3.4). Expanding the r.h.s. of (5.2) we have

$$i\epsilon * (D_\mu \Phi) = i\epsilon * (\delta_\mu \Phi) + \epsilon * (A_\mu * \Phi)\,. \qquad (5.4)$$

Equating l.h.s. and r.h.s. we see that we require

$$\begin{aligned}
(\delta A_\mu) * \Phi &= (\delta_\mu \epsilon) * \Phi - iA_\mu * (\epsilon * \Phi) + i\epsilon * (A_\mu * \Phi) \\
&= (\delta_\mu \epsilon) * \Phi - i(\epsilon * \Phi) * A_\mu + i\epsilon * (\Phi * A_\mu)\,. \qquad (5.5)
\end{aligned}$$

In the second line we took advantage of the commutativity of the star product to rewrite the r.h.s. Now the additional term in the variation of the gauge field is an associator. The associator for 3 general elements Φ_1, Φ_2, Φ_3 in $\mathcal{B}_n^*(R^{2k+1})$, denoted as $[\Phi_1, \Phi_2, \Phi_3]$, is defined as

$$[\Phi_1, \Phi_2, \Phi_3] = (\Phi_1 * \Phi_2) * \Phi_3 - \Phi_1 * (\Phi_2 * \Phi_3)\,. \qquad (5.6)$$

We introduce an operator E depending on Φ_1, Φ_3 which acts on Φ_2 to give the associator. It has an expansion in powers of derivatives.

$$\begin{aligned}
[\Phi_1, \Phi_2, \Phi_3] &= E(\Phi_1, \Phi_3)\Phi_2 \\
&= E_{\alpha_1}(\Phi_1, \Phi_3) * \delta_{\alpha_1} \Phi_2 + E_{\alpha_1 \alpha_2}(\Phi_1, \Phi_3) * \delta_{\alpha_1}\delta_{\alpha_2} \Phi_2 + \cdots\,. \qquad (5.7)
\end{aligned}$$

The explicit form of the coefficients in the expansion is worked out in appendix B, where we also define an operator $F(\Phi_1, \Phi_2)$ related to the associator as $[\Phi_1, \Phi_2, \Phi_3] = F(\Phi_1, \Phi_2)\Phi_3$.

The gauge transformation of A_μ can now be written as

$$\begin{aligned}
(\delta A_\mu) * \Phi &= (\delta_\mu \epsilon) * \Phi - i[\epsilon, \Phi, A_\mu] \\
(\delta A_\mu) &= (\delta_\mu \epsilon) - iE(\epsilon, A_\mu) \\
(\delta A_\mu) &= (\delta_\mu \epsilon) - iE_{\alpha_1}(\epsilon, A_\mu) * \delta_{\alpha_1} - iE_{\alpha_1 \alpha_2}(\epsilon, A_\mu) * \delta_{\alpha_1}\delta_{\alpha_2} + \cdots\,. \qquad (5.8)
\end{aligned}$$

This leads to a surprise. We cannot in general restrict A_μ to be a function of the Z's which generate $\mathcal{B}_n^*(R^{2k+1})$. Even if we start with such gauge fields, the gauge transformation will cause a change by $E(\epsilon, A_\mu)$ which is an operator that acts on $\mathcal{B}_n^*(R^{2k+1})$, but is not an operator of multiplication by an element of $\mathcal{B}_n^*(R^{2k+1})$ as the first term is. Instead it involves the derivatives δ_α.

This means that we should generalize the gauge field A_μ to $\hat{A}_\mu$ which should be understood as having an expansion, where A_μ is just the first term.

$$\hat{A}_\mu = A_\mu(Z) + A_{\mu\alpha_1}(Z) * \delta_{\alpha_1} + A_{\mu\alpha_1\alpha_2}(Z) * \delta_{\alpha_1}\delta_{\alpha_2} + \cdots + A_{\mu\alpha_1\cdots\alpha_n} * \delta_{\alpha_1}\delta_{\alpha_2}\cdots\delta_{\alpha_n}\,. \quad (5.9)$$

Now if we enlarge the configuration space of gauge fields, it is also natural to enlarge the configuration space of gauge parameters, introducing $\hat{\epsilon}$

$$\hat{\epsilon} = \epsilon(Z) + \epsilon_{\alpha_1}(Z) * \delta_{\alpha_1} + \epsilon_{\alpha_1\alpha_2}(Z) * \delta_{\alpha_1}\delta_{\alpha_2} + \cdots + \epsilon_{\alpha_1\alpha_2\cdots\alpha_n} * \delta_{\alpha_1}\delta_{\alpha_2}\cdots\delta_{\alpha_n}\,. \quad (5.10)$$

This will allow us the possibility that, after an appropriate gauge fixing, we recover familiar gauge fields which are functions of the coordinates only and not coordinates and derivatives.

On physical grounds, given that the non-associative $2k$-sphere $\mathcal{A}_n(S^{2k})$ arises as the algebra describing the base space of field theory for spherical brane worldvolumes [2, 3, 27, 5, 6] we expect that gauge theory on these algebras, and the related algebras $\mathcal{A}^*_n(R^{2k+1})$, $\mathcal{B}^*_n(R^{2k+1})$, $\mathcal{B}^*_n(S^{2k})$ should exist. We also know that they must approach ordinary gauge theory in the large n limit. It is also reasonable to expect that the large n limit is approached smoothly. Hence we expect that gauge fields which are functions of coordinates must be a valid way to describe the configuration space at finite n. The surprise which comes from exploring the finite n structure of the gauge transformations is that such a configuration space is only a partially gauge fixed description.

We write out the gauge transformations of $\hat{A}_\mu$ which follow from covariance of D_μ when we keep the first term in the derivative expansion of the generalized gauge transformation $\hat{\epsilon}$ and the first term in the derivative expansion of $\hat{A}_\mu$. The two operators E and F related to the associator, which we introduced in appendix B, are both useful.

$$\begin{aligned}
\delta A_\mu = {} & [\delta_\mu\epsilon + iA_{\mu\alpha_1} * (\delta_{\alpha_1}\epsilon) - i\epsilon_{\alpha_1} * (\delta_{\alpha_1}A_\mu) + \cdots] \times \\
& \times [-iE_{\beta_1}(\epsilon, A_\mu) * \delta_{\beta_1} + iF_{\beta_1}(A_{\mu\alpha_1}, \delta_{\alpha_1}\epsilon) * \delta_{\beta_1} - iF_{\beta_1}(\epsilon_{\alpha_1}, \delta_{\alpha_1}A_\mu) * \delta_{\beta_1} + \cdots] + \\
& + [-iE_{\beta_1\beta_2}(\epsilon, A_\mu) * \delta_{\beta_1}\delta_{\beta_2} - iE_{\beta_1}(\epsilon, A_{\mu\alpha_1}) * \delta_{\beta_1}\delta_{\alpha_1} + iF_{\beta_1\beta_2}(A_{\mu\alpha_1}, \delta_{\alpha_1}\epsilon) * \delta_{\beta_1}\delta_{\beta_2} - \\
& \quad - iF_{\beta_1\beta_2}(\epsilon_{\alpha_1}, \delta_{\alpha_1}A_\mu) * \delta_{\beta_1}\delta_{\beta_2} + \cdots]\,.
\end{aligned} \quad (5.11)$$

We have separated the terms which involve functions, from those involving one derivative operators, and two-derivative operators etc. We have not exhibited terms involving higher derivative transformations of A_μ, but it should be possible to exhibit them more explicitly using the general formulae for E and F operators derived in appendix B. If we set the derivative parts in A_μ to zero, e.g. $A_{\mu\alpha_1} = 0$, (5.11) show that the purely coordinate dependent pieces still get modified from the usual $\delta_\mu\epsilon$, and from the requirement that the first and second derivative corrections to A_μ vanish, we get

$$\begin{aligned}
-iE_{\beta_1}(\epsilon, A_\mu) - iF_{\beta_1}(\epsilon_{\alpha_1}, \delta_{\alpha_1}A_\mu) + \cdots &= 0 \\
-iE_{\beta_1\beta_2}(\epsilon, A_\mu) - iF_{\beta_1\beta_2}(\epsilon_{\alpha_1}, \delta_{\alpha_1}A_\mu) + \cdots &= 0\,.
\end{aligned} \quad (5.12)$$

For finite n the terms left as dotted should be expressible as a finite sum since powers of derivatives higher than the n'th automatically vanish on any element of the algebra $\mathcal{B}^*_n(R^{2k+1})$.

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5.2 Formula for the associator

We consider the associator for a triple of general elements $Z_{\mu(S)}$, $Z_{\mu(T)}$, $Z_{\mu(V)}$ where S, T, V are sets of distinct positive integers. We use (4.19) for $Z_{\mu(S)} * Z_{\mu(T)}$. Then we compute $(Z_{\mu(S)} * Z_{\mu(T)}) * Z_{\mu(V)}$ as

$$\begin{aligned}(Z_{\mu(S)} * Z_{\mu(T)}) * Z_{\mu(V)} = & \sum_{|S_1|,|S_2|} \sum_{|S_3|,|S_4|} \delta(|S_1|,|S_2|)\delta(|S_3|,|S_4|) \times \\ & \times \sum_{S_1 \subset S} \sum_{S_2 \subset T} \sum_{S_3 \subset \{S \cup T\}} \sum_{S_4 \subset V} \frac{1}{n^{2|S_1|+2|S_3|}} \frac{n!}{(n-|S_1|)!} \frac{n!}{(n-|S_3|)!} \times \\ & \times \tilde{\delta}(\mu(S_1),\mu(S_2))\tilde{\delta}(\mu(S_3),\mu(S_4)) \times \\ & \times Z_{\mu(\{S\cup T\}\setminus\{S_1\cup S_2\cup S_3\cup S_4\})} \,. \end{aligned} \tag{5.13}$$

It is useful to separate the sum over subsets S_3 and S_4 to distinguish the delta functions δ_{SV} between indices belonging to the sets S and V and delta functions δ_{TV} between indices belonging to the sets T and V. We decompose the cardinalities of the sets

$$\begin{aligned} |S_3| &= |S_3^{(1)}| + |S_3^{(2)}| \\ |S_4| &= |S_4^{(1)}| + |S_4^{(2)}| \end{aligned} \tag{5.14}$$

and the sets as

$$S_3 = S_3^{(1)} \cup S_3^{(2)} \,,$$

where

$$\begin{aligned} S_3^{(1)} &\subset S \setminus S_1 \qquad S_3^{(2)} \subset T \setminus S_2 \\ S_4 &= S_4^{(1)} \cup S_4^{(2)} \end{aligned}$$

where

$$S_4^{(1)} \subset V \qquad S_4^{(2)} \subset V \,. \tag{5.15}$$

The indices in the set $S_3^{(1)}$ are paired with those in the set $S_4^{(1)}$ and likewise those in $S_3^{(2)}$ are paired with $S_4^{(2)}$, so that we have $|S_3^{(1)}| = |S_4^{(1)}|$ and $|S_3^{(2)}| = |S_4^{(2)}|$. We express the product in (5.13)

$$\begin{aligned}(Z_{\mu(S)} * Z_{\mu(T)}) * Z_{\mu(V)} = & \\ = \sum_{|S_1|,|S_2|} \sum_{|S_3^{(1)}|,|S_4^{(1)}|} \sum_{|S_3^{(2)}|,|S_4^{(2)}|} & \delta(|S_1|,|S_2|)\delta\left(|S_3^{(1)}|,|S_4^{(1)}|\right)\delta(|S_3^{(2)}|,|S_4^{(2)}|) \times \\ \times \sum_{S_1\subset S, S_2 \subset T} \sum_{S_3^{(1)} \subset \{S\setminus S_1\}} & \sum_{S_3^{(2)} \subset \{T \setminus S_2\}} \sum_{S_4^{(1)} \subset V} \sum_{S_4^{(2)} \subset V} \tilde{\delta}(\mu(S_1),\mu(S_2)) \times \\ \times \tilde{\delta}(\mu(S_3^{(1)}),\mu(S_4^{(1)}))&\tilde{\delta}(\mu(S_3^{(2)}),\mu(S_4^{(2)})) Z_{\mu(\{S\cup T\}\setminus\{S_1\cup S_2\cup S_3\cup S_4\})} \times \\ \times \frac{1}{n^{2|S_1|+2|S_2|}} \frac{n!}{(n-|S_1|)!} & \frac{n!}{(n-|S_3^{(1)}|-|S_3^{(2)}|)!} \,. \end{aligned} \tag{5.16}$$

Note that the combinatoric factor needed to rewrite the $\tilde{\delta}(\mu(S_3),\mu(S_4))$ in terms of the sum of products $\tilde{\delta}(\mu(S_3^{(1)}),\mu(S_4^{(1)}))\tilde{\delta}(\mu(S_3^{(2)}),\mu(S_4^{(2)}))$ is 1. The expression $\tilde{\delta}(\mu(S_3),\mu(S_4))$ contains $|S_3|!$ Kronecker delta's. When we sum the product of two $\tilde{\delta}$'s we have $\binom{|S_4|}{|S_4^{(1)}|} = \binom{|S_3|}{|S_3^{(1)}|}$ as the number of ways of picking sets $S_4^{(1)}$ from S_4. When this multiplies the $|S_3^{(1)}|!|S_3^{(2)}|!$ for the number of Kronecker δ's in the product of two $\tilde{\delta}$'s we get exactly $|S_3|!$. This explains why the combinatoric factor is 1.

In a similar way we can write $Z_{\mu(S)} * (Z_{\mu(T)} * Z_{\mu(V)})$. After separating the $\tilde{\delta}$ from the second multiplication into a product of two $\tilde{\delta}$'s containing pairings of S with V and pairings of S with T, we have an expression of the same form above except that the factor $\frac{n!}{(n-|S_1|)!}\frac{n!}{(n-|S_3^{(1)}|-|S_3^{(2)}|)!}$ is replaced by $\frac{n!}{(n-|S_3^{(2)}|)!}\frac{n!}{(n-|S_1|-|S_3^{(1)}|)!}$. Hence we can write down an expression for the associator $(Z_{\mu(S)} * Z_{\mu(T)}) * Z_{\mu(V)} - Z_{\mu(S)} * (Z_{\mu(T)} * Z_{\mu(V)})$ as a sum of the above form with a coefficient which is a difference of these two factors. It is instructive to rewrite the final outcome of the multiplications and the associator using a notation for the summation labels which is more informative and less prejudiced by the order in which the multiplication was done. We write R_{ST} for the subset of S which is paired with indices in T, and R_{TS} for the subset of indices in T which are paired with indices in S. The cardinalities are denoted as usual by $|R_{ST}| = |R_{TS}|$. Similarly we use subsets R_{SV}, R_{VS} for pairings between the sets S and V, and the subsets R_{TV}, R_{VT}. This way we can write the associator as

$$\begin{aligned}
&(Z_{\mu(S)} * Z_{\mu(T)}) * Z_{\mu(V)} - Z_{\mu(S)} * (Z_{\mu(T)} * Z_{\mu(V)}) = \\
&= \sum_{R_{ST},R_{TS}} \sum_{R_{SV},R_{VS}} \sum_{R_{TV},R_{VT}} \tilde{\delta}(\mu(R_{ST}),\mu(R_{TS}))\tilde{\delta}(\mu(R_{SV}),\mu(R_{VS})) \times \\
&\quad \times \frac{\tilde{\delta}(\mu(R_{TV}),\mu(R_{VT}))}{n^{2|R_{ST}|+2|R_{SV}|+2|R_{TV}|}} Z_{\mu(\ \{\ S\cup T\cup V\}\setminus\{R_{ST}\cup R_{TS}\cup R_{SV}\cup R_{VS}\cup R_{TV}\cup R_{VT}\ \}\ )} \times \\
&\quad \times \left(\frac{n!}{(n-|R_{ST}|)!}\frac{n!}{(n-|R_{SV}|-|R_{VT}|)!} - \frac{n!}{(n-|R_{TV}|)!}\frac{n!}{(n-|R_{ST}|-|R_{SV}|)!}\right).
\end{aligned} \tag{5.17}$$

Let us define $A(|R_{ST}|,|R_{SV}|,|R_{TV}|)$ to be the coefficient in the expression above

$$\begin{aligned}
&A(|R_{ST}|,|R_{SV}|,|R_{TV}|) \equiv \\
&\quad \equiv \frac{n!}{(n-|R_{ST}|)!}\frac{n!}{(n-|R_{SV}|-|R_{VT}|)!} - \frac{n!}{(n-|R_{TV}|)!}\frac{n!}{(n-|R_{ST}|-|R_{SV}|)!}
\end{aligned} \tag{5.18}$$

We observe that exchanging the S and V labels in A is an antisymmetry

$$A(|R_{VT}|,|R_{VS}|,|R_{ST}|) = -A(|R_{ST}|,|R_{SV}|,|R_{TV}|)\,. \tag{5.19}$$

This is to be expected and is nice check on the above formulae. Indeed for a *commutative* algebra as we have here

$$\begin{aligned}
[Z_{\mu(S)},Z_{\mu(T)},Z_{\mu(V)}] &= (Z_{\mu(S)} * Z_{\mu(T)}) * Z_{\mu(V)} - Z_{\mu(S)} * (Z_{\mu(T)} * Z_{\mu(V)}) \\
&= Z_{\mu(V)} * (Z_{\mu(S)} * Z_{\mu(T)}) - (Z_{\mu(T)} * Z_{\mu(V)}) * Z_{\mu(S)} \\
&= Z_{\mu(V)} * (Z_{\mu(T)} * Z_{\mu(S)}) - (Z_{\mu(V)} * Z_{\mu(T)}) * Z_{\mu(S)} \\
&= -[Z_{\mu(V)},Z_{\mu(T)},Z_{\mu(S)}]\,.
\end{aligned} \tag{5.20}$$

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5.3 Simple examples of associator

As another check on the general formula (5.17) we consider an example of the form $[Z_{\mu(S)}, Z_{\mu_1}, Z_{\mu_2}]$.

$$\begin{aligned}
(Z_{\mu(S)} * Z_{\mu_1}) &= Z_{\mu_1\mu(S)} + \sum_{i\in S}\frac{1}{n}\delta_{\mu_1\mu_i}Z_{\mu(S\backslash i)} \\
(Z_{\mu(S)} * Z_{\mu_1}) * Z_{\mu_2} &= Z_{\mu_1\mu_2\mu(S)} + \frac{\delta_{\mu_1\mu_2}}{n}Z_{\mu(S)} + \sum_{i\in S}\frac{1}{n}\delta_{\mu_2\mu_i}Z_{\mu_1\mu(S\backslash i)} + \\
&\quad + \sum_{i,j\in S}\frac{1}{n^2}\delta_{\mu_1\mu_i}\delta_{\mu_2\mu_j}Z_{\mu(S\backslash i,j)}\,. \qquad (5.21)
\end{aligned}$$

When we multiply in the other order

$$\begin{aligned}
Z_{\mu_1} * Z_{\mu_2} &= Z_{\mu_1\mu_2} + \frac{\delta_{\mu_1\mu_2}}{n} \\
Z_{\mu(S)} * (Z_{\mu_1} * Z_{\mu_2}) &= Z_{\mu_1\mu_2\mu(S)} + \frac{1}{n}\sum_{i\in S}\delta_{\mu_1\mu_i}Z_{\mu_2\mu(S\backslash i)} + \frac{1}{n}\sum_{i\in S}\delta_{\mu_2\mu_i}Z_{\mu_1\mu(S\backslash i)} + \\
&\quad + \frac{n(n-1)}{n^4}\sum_{i,j\in S}\delta_{\mu_1\mu_i}\delta_{\mu_2\mu_j}Z_{\mu(S\backslash i,j)} + \frac{\delta_{\mu_1\mu_2}}{n}Z_{\mu(S)}\,. \qquad (5.22)
\end{aligned}$$

Hence the associator is

$$[Z_{\mu(S)}, \mu_1, \mu_2] = (Z_{\mu(S)} * Z_{\mu_1}) * Z_{\mu_2} - Z_{\mu(S)} * (Z_{\mu_1} * Z_{\mu_2}) = \frac{1}{n^3}\sum_{i,j\in S}\delta_{\mu_1\mu_i}\delta_{\mu_2\mu_j}Z_{\mu(S\backslash i,j)}\,. \quad (5.23)$$

Note that if $|S| = 1$ the associator is zero. The single δ terms cancel in the associator. This follows from (5.17) because $A(1,0,0) = A(0,1,0) = A(0,0,1) = 0$. The coefficient $\frac{1}{n^3}$ of the two δ terms can be read off from (5.17) by noting that for $|R_{ST}| = |R_{SV}| = 1$ and $|R_{TV}| = 0$, we have

$$\frac{1}{n^{2|R_{ST}|+2|R_{SV}|+2|R_{TV}|}}A(|R_{ST}|, |R_{SV}|, |R_{TV}|) = \frac{1}{n^4}[n^2 - n(n-1)] = \frac{1}{n^3}\,. \qquad (5.24)$$

The expressions developed above can lead to the construction of operators E and F related to the associator. The operators are described in appendix B and used in the first part of this section.

6. Gauge theory on $\mathcal{B}_n^*(S^{2k})$

We will obtain an action for the deformed algebra of functions $\mathcal{B}_n^*(S^{2k})$ which is obtained by applying the constraint of constant radius to the algebra $\mathcal{B}_n^*(R^{2k+1})$.

6.1 A finite n action which approaches the abelian Yang-Mills action on S^{2k}

We begin by writing the ordinary pure Yang Mills action on S^{2k} in terms of the embedding coordinates in R^{2k+1}. We describe the S^{2k} in terms of $\sum_\mu z_\mu z_\mu = 1$. The derivatives $\frac{\partial}{\partial z_\mu}$

can be expanded into an angular part and a radial part.

$$\begin{aligned}\frac{\partial}{\partial z_\mu} &= \frac{\partial \theta^a}{\partial z_\mu}\frac{\partial}{\partial \theta^a} + \frac{\partial r}{\partial z_\mu}\frac{\partial}{\partial r} \\ &= P_\mu + \frac{z_\mu}{r}\frac{\partial}{\partial r}\,. \end{aligned} \tag{6.1}$$

We have denoted as $P_\mu = P^a_\mu \frac{\partial}{\partial \theta^a}$ the projection of the derivative along the sphere. Consider the commutator $[P_\mu - iA_\mu, P_\nu - iA_\nu]$ for gauge fields A_μ satisfying $z_\mu A_\mu = 0$ and which are functions of the angular variables only. A general gauge field A_μ has an expansion $A_\mu = P^a_\mu A_a + \frac{z_\mu}{r} A_r$. The condition $z_\mu A_\mu = 0$ guarantees that $A_r = 0$, hence

$$A_\mu = P^a_\mu A_a\,. \tag{6.2}$$

The following are useful observations

$$\begin{aligned}[D_\mu, D_\nu] &= [P_\mu - iA_\mu, P_\nu - iA_\nu] \\ &= P^a_\mu P^b_\nu F_{ab} + L^a_{\mu\nu} D_a \\ &= P^a_\mu P^b_\nu F_{ab} + z_\mu D_\nu - z_\nu D_\mu\,. \end{aligned} \tag{6.3}$$

We can then express

$$P^a_\mu P^b_\nu F_{ab} = [D_\mu, D_\nu] - (z_\mu D_\nu - z_\nu D_\mu)\,. \tag{6.4}$$

Now observe that the $P^a_\mu P^b_\mu = G^{ab}$, i.e the inverse of the metric induced on the sphere by its embedding in R^{2k+1}, which is also the standard round metric. Hence we can write the Yang Mills action as

$$\int d^4\theta \sqrt{G} G^{ac} G^{bd} F_{ac} F_{bd} = \int d^4\theta \sqrt{G} ([D_\mu, D_\nu] - (z_\mu D_\nu - z_\nu D_\mu))^2\,. \tag{6.5}$$

We define a commutative non-associative $\mathcal{B}^*_n(S^{2k+1})$ by imposing conditions $Z_{\alpha\alpha} = \frac{(n-1)}{n} R^2$. Z operators with more indices contracted e.g. $Z_{\alpha\alpha\beta\beta}$ can be deduced because they appear in products like $Z_{\alpha\alpha} * Z_{\beta\beta}$ and their generalizations. We have introduced an extra generator R which corresponds to the radial coordinate transverse to the sphere.

The projection operators P_α obey $P_\alpha = \delta_\alpha - Q_\alpha$. We have already shown that δ_α acts as a derivation of the star product. The behaviour of Q_α can be obtained similarly. In general we can expect it to be a deformed derivation. In classical geometry :

$$\begin{aligned}\delta_\alpha R &= \frac{Z_\alpha}{R} \\ Q_\alpha R &= \frac{Z_\alpha}{R} \\ Q_\alpha Z_\beta &= \frac{Z_\alpha Z_\beta}{R^2}\,. \end{aligned} \tag{6.6}$$

This suggests that the definitions at finite n :

$$\begin{aligned}\delta_\alpha R &= \frac{Z_\alpha}{R} \\ Q_\alpha R &= \frac{Z_\alpha}{R} \\ Q_\alpha Z_{\mu(S)} &= \frac{|S|}{R^2} Z_\alpha * Z_{\mu(S)}\,. \end{aligned} \tag{6.7}$$

In the classical case Q_α is a derivation so its action on a polynomial in the Z's is defined by knowing its action on the Z's and using Leibniz rule. In the finite n case we have given the action on a general element of the algebra and we leave the deformation of the Leibniz rule to be computed. The deformed Leibniz rule can be derived by techniques similar to the ones used in section 3. Following section 5, we can determine the gauge transformation of A_μ by requiring covariance of $D_\mu = P_\mu - iA_\mu$. The deformation of Leibniz rule implies that the gauge transformation law will pick up extra terms due to

$$\begin{aligned} P_\alpha(\epsilon * \Phi) &= (\delta_\alpha - Q_\alpha)(\epsilon * \Phi) \\ &= (\delta_\alpha \epsilon) * (\Phi) + \epsilon * (\delta_\alpha \Phi) - (Q_\alpha \epsilon) * \Phi - \epsilon * (Q_\alpha \Phi) - (Q_\alpha^{(1)} \epsilon) * (Q_\alpha^{(2)} \Phi) + \cdots \end{aligned} \tag{6.8}$$

The last term is a schematic indication of the form we may expect for the corrections to the Leibniz rule, following section 2. More details on the deformed Leibniz rule for Q_α are given in appendix C.

As in section 5, the non-zero associator will require generalizing the gauge parameters and gauge fields in order to make the gauge invariance manifest. It is not hard to write down a finite n action (without manifest gauge invariance) which reproduces the classical action on the sphere

$$\int ([P_\mu - iA_\mu, P_\nu - iA_\nu] - (Z_\mu * D_\nu - Z_\nu * D_\mu)]^2 \,. \tag{6.9}$$

It has the same form as (6.5). The A_μ are chosen to satisfy $Z_\mu * A_\mu = 0$ to guarantee they have components tangential to the sphere, and they are expanded in symmetric traceless polynomials in the algebra $\mathcal{B}_n^*(S^{2k})$. The integral is defined to pick out the coefficient of the integrand which is proportional to the identity function in the algebra $\mathcal{B}_n(S^{2k})$. It is reasonable to guess that this action can be obtained as a gauge fixed form of an action where the gauge invariances are manifest at finite n. The equations to be solved now involve the operators E and F as well as the additional terms due to the corrections to Leibniz rule for Q_α.

7. Outlook

7.1 The non-abelian generalization and matrix brane constructions

In the bulk of this paper we have considered abelian theories on a class of commutative but non-associative algebras related to the fuzzy $2k$-spheres. It is also possible to generalize the calculations to non-abelian theories. Indeed these are the kinds of theories that show up in brane constructions [2, 3, 27]. In this case we need to consider covariant derivatives of the form $\delta_\mu - iA_\mu$ where the A_μ are matrices whose entries take values in the $\mathcal{A}_n(S^{2k})$. The Φ are also Matrices containing entries which take values in the algebra. Again we define the transformation of $\delta\Phi = \epsilon * \Phi$ or writing out to include the Matrix indices $\delta\Phi_{ij} = \epsilon_{ik} * \Phi_{kj}$. As in the abelian case, we can construct the gauge transformation law for A_μ by requiring covariance of $P_\mu - iA_\mu$. It will very interesting to construct these non-abelian generalizations in detail. They would answer a very natural question which arises in the D-brane applications where one constructs a higher brane worldvolume from a lower brane worldvolume. There is evidence from D-brane physics that the fuzzy sphere ansatze on the lower brane worldvolume leads to physics equivalent to that described by the higher brane worldvolume with a background non-abelian field strength. Following the logic of [11] it should be possible to start with the lower brane worldvolume and derive an action for the fluctuations which is a non-abelian theory on the higher dimensional sphere. Thus the evidence gathered for the agreement of the physical picture (brane charges, energies etc.) from lower brane and higher brane worldvolumes would be extended into a derivation of the higher brane action from the lower brane action. Since the only algebra structure we can put on the truncated spectrum of fuzzy spherical harmonics at finite n is a non-associative one [5], the techniques of this paper provide an avenue for constructing this non-abelian theory.

An alternative description of the fluctuations in these Matrix brane constructions is in terms of an abelian theory on a higher dimensional geometry $\mathrm{SO}(2k+1)/\,\mathrm{U}(k)$ [6, 28]. Combining this with the physical picture discussed above of a non-abelian theory on the commutative/non-associative sphere, we are lead to expect that there should be a duality between the abelian theory on the $\mathrm{SO}(2k+1)/\,\mathrm{U}(k)$ and the abelian theory on S^{2k} [6]. The explicit construction of the detailed form of the non-abelian theories on the the fuzzy sphere algebras $\mathcal{A}_n(S^{2k})$ will also be interesting since it will allow us to explore the mechanisms of this duality. There has been a use of effective field theory on the higher dimensional geometry in the quantum Hall effect [29, 30].

7.2 Physical applications of the abelian theory

In applications of the fuzzy 4 sphere to the ADS/CFT as in [31][32] we have abelian gauge fields in spacetime (in addition to gravity fields etc.) so we should expect an abelian gauge theory on the fuzzy sphere if we want the fuzzy 4-sphere to be a model for spacetime. In the context of the of longitudinal 5-branes [2] in the DLCQ (Discrete Light cone quantization) of M-Theory, it was originally somewhat surprising that only spherical 5-brane numbers of the form $\frac{(n+1)(n+2)(n+3)}{6}$ for integer n could be described. We may ask for example why there is not a DLCQ description of a single spherical 5-brane. The abelian field theories considered here would be candidate worldvolume descriptions for the single 5-brane in the $DLCQ$ of M-theory, but it is not clear how to directly derive them from BFSS Matrix theory.

7.3 Other short comments

We have only begun to investigate the structure of gauge theories related to commutative, non-associative algebras. For applications to Matrix Theory fuzzy spheres, it will be interesting to work directly with $\mathcal{A}_n(R^{2k+1})$ and $\mathcal{A}_n(S^{2k})$ (along with the twisted algebras $\mathcal{A}_n^*(R^{2k+1})$ and $\mathcal{A}_n^*(S^{2k})$), instead of the $\mathcal{B}_n(R^{2k+1})$ and $\mathcal{B}_n(S^{2k})$ (along with their twisted versions $\mathcal{B}_n^*(R^{2k+1})$ $\mathcal{B}_n^*(S^{2k})$) which were introduced for simplicity, by considering a large D limit of $\mathcal{A}_n(R^{D+1})$. Instanton equations for gauge theories on these algebras will be interesting. In connecting these field theories on $\mathcal{A}_n(S^{2k})$ to the Matrix Theory fuzzy $2k$-spheres, it will be necessary to expand around a spherically symmetric instanton background. Developing gauge fixing procedures and setting up perturbation theory are other directions to be explored. The projection operators P_α together with the SO$(2k+1)$ generators $L_{\mu\nu}$ form the conformal algebra SO$(2k+1,1)$. It will be interesting to see how a finite n deformation of this acts on the finite n commutative/non-associative algebra of the fuzzy $2k$-sphere.

By replacing $\delta_{\mu\nu}$ with $\eta_{\mu\nu}$ (the SO$(2k,1)$ lorentzian invariant) in (2.10), (2.13) and (4.19) we get commutative, non-associative algebras which deform lorentzian spacetime. Since the commutator $[Z_\mu, Z_\nu]$ is zero, we do not have a $\theta_{\mu\nu}$ which breaks Lorentz invariance. If we assign standard dimensions of length to Z and introduce a dimensionful deformation parameter θ to make sure all terms in (2.13) and (4.19) are of same dimension, we only need a θ which is *scalar* under the Lorentz group. It will be very interesting to study these algebras as Lorentz invariant deformations of spacetime in the context of the transplanckian problem for example, along the lines of [33].

The generalized gauge fields and gauge parameters which we were lead to, as a consequence of formulating gauge theory for the non-associative algebras, are strongly reminiscent of constructions that have appeared in descriptions of the geometry of W-algebras [34], where higher symmetric tensor fields appear as gauge parameters and gauge fields. It is possible that insights from W-geometry can be used in further studies of gauge theories on commutative, non-associative algebras. If a connection to W-algebras can be made, the equations such as (5.12) and its generalizations, constraining the generalized gauge parameters (5.10) by requiring that the gauge fields remain purely coordinate dependent, would be conditions which define an embedding of the U(1) gauge symmetry inside a W-algebra. The embedding is trivial when the non-associativity is zero, given simply by setting the $\epsilon_{\alpha_1}, \epsilon_{\alpha_1\alpha_2} \cdots = 0$, but requires the higher tensors to be non-zero and related to each other when the non-associativity is turned on. It is also notable that generalized gauge fields involving dependence on derivatives have appeared [35] when considering non-associative algebras related to non-zero background H fields [36]. Further geometrical understanding of these features appears likely to involve the BV formalism [37].

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A. The classical product in terms of the star product m_2^*

We introduce an algebra $\mathcal{B}_n^c(R^{2k+1})$ with product m_2^c which, as a vector space, is the same as $\mathcal{B}_n(R^{2k+1})$ and $\mathcal{B}_n^*(R^{2k+1})$. The product resembles the classical product in that it is just concatenates the indices on the Z's.

$$m_2^c(Z_{\mu(S)} \otimes Z_{\mu(T)}) = Z_{\mu(S\cup T)}\,. \tag{A.1}$$

The formula (4.19) for the product m_2^* can be rewritten in terms of m_2^c

$$m_2^* = \sum_{k=0}^{n} \frac{1}{n^{2k}k!}\frac{n!}{(n-k)!} \sum_{\alpha_1\cdots\alpha_k} m_2^c \cdot (\delta_{\alpha_1}\cdots\delta_{\alpha_k} \otimes \delta_{\alpha_1}\cdots\delta_{\alpha_k})\,. \tag{A.2}$$

It is very useful to have the inverse formula where m_2^c is expressed in terms of m_2^*. This can be used to write general formulae for the associator by manipulating (5.17).

$$m_2^c = \sum_{k=0}^{n} \frac{(-1)^k}{n^{2k-1}k!}\frac{(n+k-1)!}{n!} \sum_{\alpha_1\cdots\alpha_k} m_2^* \cdot (\delta_{\alpha_1}\cdots\delta_{\alpha_k} \otimes \delta_{\alpha_1}\cdots\delta_{\alpha_k})\,. \tag{A.3}$$

Proving this inversion formula makes use of a combinatoric identity. Let us define $f_a(n) = \frac{n!}{(n-a)!}\frac{1}{n^{2a}}$, and $\tilde{f}_a(n) = \frac{(-1)^a}{n^{2a-1}}\frac{(n+a-1)!}{n!}$. The combinatoric identity is

$$\tilde{f}_k(n) = \sum_{l=1}^{k}(-1)^l \sum_{a_1,a_2,\cdots a_l\geq 1:\sum a_i=k} f_{a_1}f_{a_2}\cdots f_{a_l}\frac{k!}{a_1!a_2!\cdots a_l!}\cdot\frac{l!}{|S(a_1,a_2,\cdots a_l)|}\,. \tag{A.4}$$

We are summing over ways of writing the positive integer k as a sum of integers $a_i \geq 1$. For each such splitting of k, the symmetry factor $S(a_1, a_2, \cdots a_l)$ is the product $n_1!n_2!\cdots$ where n_1 is the number of 1's, n_2 the number of 2's etc. We have checked the identity for k up to 6 using Maple and find the simple form of $\tilde{f}$ given above. An analytic proof for general k would be desirable.

B. Operators related to the associator

In (5.17) we have written down the associator of three general elements of the algebra $\mathcal{B}_n^*(R^{2k+1})$ in terms of $Z_{\mu(\{S\cup T\cup V\}\backslash\{R_{ST}\cup R_{TS}\cup R_{SV}\cup R_{VS}\cup R_{TV}\cup R_{VT}\})}$. This can be expressed in terms of the product m_2^c

$$\begin{aligned}
&Z_{\mu(\ \{\ S\cup T\cup V\}\backslash\{R_{ST}\cup R_{TS}\cup R_{SV}\cup R_{VS}\cup R_{TV}\cup R_{VT}\ \}\ )} = \\
&\quad = m_2^c\cdot(m_2^c\otimes 1)\cdot(Z_{\mu(S\backslash\{R_{ST}\cup R_{SV}\})}\otimes Z_{\mu(T\backslash\{R_{TS}\cup R_{TV}\})}\otimes Z_{\mu(V\backslash\{R_{VS}\cup R_{VT}\})})\\
&\quad = m_2^c\cdot(m_2^c\otimes 1)\cdot(Z_{\mu(S\backslash\{R_{ST}\cup R_{SV}\})}\otimes Z_{\mu(V\backslash\{R_{VS}\cup R_{VT}\})}\otimes Z_{\mu(T\backslash\{R_{TS}\cup R_{TV}\})})\,.
\end{aligned} \tag{B.1}$$

By using (5.17) together with (B.1) and converting m_2^c to m_2^* we can get formulas for the associator in terms of m_2^*. We

$$\begin{aligned}
&[\Phi_1,\Phi_2,\Phi_3] = \\
&= \sum_{r_1=0}^{n}\sum_{r_2=0}^{n}\sum_{r_3=0}^{n}\sum_{a_1\cdots a_{r_1}}\sum_{b_1\cdots b_{r_2}}\sum_{c_1\cdots c_{r_3}}\sum_{m=0}^{n}\sum_{\beta_1\cdots\beta_m}\sum_{l=0}^{n}\sum_{\alpha_1\cdots\alpha_l}\frac{\tilde{f}_l(n)}{l!}\times\\
&\quad\times\frac{\tilde{f}_m(n)}{m!}\frac{1}{r_1!r_2!r_3!}[f_{r_1}(n)f_{r_2+r_3}(n)-f_{r_3}(n)f_{r_1+r_2}(n)]\times\\
&\quad\times\ \delta_{\beta_1\cdots\beta_m}(\delta_{\alpha_1}\cdots\delta_{\alpha_l}\delta_{a_1}\cdots\delta_{a_{r_1}}\delta_{b_1}\cdots\delta_{b_{r_2}}\Phi_1 * \delta_{\alpha_1}\cdots\alpha_l\delta_{a_1}\cdots\delta_{a_{r_1}}\delta_{c_1}\cdots\delta_{c_{r_3}}\Phi_2) *\\
&\quad * \delta_{\beta_1}\cdots\delta_{\beta_m}\delta_{b_1}\cdots\delta_{b_{r_2}}\delta_{c_1}\cdots\delta_{c_{r_3}}\Phi_3\,.
\end{aligned} \tag{B.2}$$

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The a, b, c, α, β indices all run from 1 to $2k+1$. This expresses the associator $[\Phi_1, \Phi_2, \Phi_3]$ as an operator F depending on Φ_1, Φ_2 and acting on Φ_3

$$[\Phi_1, \Phi_2, \Phi_3] = F_{\alpha_1}(\Phi_1, \Phi_2) * \delta_{\alpha_1}\Phi_3 + F_{\alpha_1\alpha_2}(\Phi_1, \Phi_2) * \delta_{\alpha_1}\delta_{\alpha_2}\Phi_3 + \cdots . \tag{B.3}$$

The successive terms are subleading in the $1/n$ expansion.

We can also use the second line of (B.2) to write the associator as

$$\begin{aligned}
&[\Phi_1, \Phi_2, \Phi_3] = \\
&= \sum_{r_1=0}^{n} \sum_{r_2=0}^{n} \sum_{r_3=0}^{n} \sum_{a_1\cdots a_{r_1}} \sum_{b_1\cdots b_{r_2}} \sum_{c_1\cdots c_{r_3}} \sum_{m=0}^{n} \sum_{\beta_1\cdots\beta_m} \sum_{l=0}^{n} \sum_{\alpha_1\cdots\alpha_l} \frac{\tilde{f}_l(n)}{l!} \times \\
&\times \frac{\tilde{f}_m(n)}{m!} \frac{1}{r_1! r_2! r_3!} [f_{r_1}(n) f_{r_2+r_3}(n) - f_{r_3}(n) f_{r_1+r_2}(n)] \times \\
&\times \delta_{\beta_1\cdots\beta_m} (\delta_{\alpha_1} \cdots \alpha_l \delta_{a_1} \cdots \delta_{a_{r_1}} \delta_{b_1} \cdots \delta_{b_{r_2}} \Phi_1 * \delta_{\alpha_1} \cdots \alpha_l \delta_{b_1} \cdots \delta_{b_{r_2}} \delta_{c_1} \cdots \delta_{c_{r_3}} \Phi_3) * \\
&* \delta_{\beta_1} \cdots \beta_m \delta_{b_1} \cdots \delta_{b_{r_1}} \delta_{a_1} \cdots \delta_{a_{r_1}} \Phi_2 .
\end{aligned} \tag{B.4}$$

This gives the associator as an operator E depending on Φ_1, Φ_3 and acting on Φ_2

$$[\Phi_1, \Phi_2, \Phi_3] = E_{\alpha_1}(\Phi_1, \Phi_3) * \delta_{\alpha_1}\Phi_3 + E_{\alpha_1\alpha_2}(\Phi_1, \Phi_3) * \delta_{\alpha_1}\delta_{\alpha_2}\Phi_2 + \cdots . \tag{B.5}$$

The successive terms are subleading in the $1/n$ expansion.

C. Deformed Leibniz rule for Q_α

Using the definition of Q_α we can work out the deformation of its Leibniz rule.

$$\begin{aligned}
&Q_\alpha \left(Z_{\mu(S)} * Z_{\mu(T)}\right) - \left(Q_\alpha Z_{\mu(S)}\right) * Z_{\mu(T)} - Z_{\mu(S)} * \left(Q_\alpha Z_{\mu(T)}\right) = \\
&= -\frac{1}{R^2} \sum_{T_1\subset S} \sum_{T_2\subset T} \tilde{\delta}(\mu(T_1), \mu(T_2)) \frac{2|T_1|}{n^{2|T_1|}} \frac{n!}{(n-|T_1|)!} Z_{\alpha\mu(S\cup T\setminus\{T_1\cup T_2\})} + \\
&+ \frac{1}{R^2} \sum_{T_1\subset S} \sum_{i,T_2\subset T} \left(\frac{-2|T_1|}{n} + \frac{|T_1||S|}{n}\right) \frac{n!}{n^{2|T_1|}(n-|T_1|)!} \times \\
&\times \delta_{\alpha\mu_i} \tilde{\delta}(\mu(T_1), \mu(T_2)) Z_{\mu(S\cup T\setminus\{T_1\cup T_2\})} + \\
&+ \frac{1}{R^2} \sum_{i,T_1\subset S} \sum_{T_2\subset T} \left(\frac{-2|T_1|}{n} + \frac{|T_1||T|}{n}\right) \frac{n!}{n^{2|T_1|}(n-|T_1|)!} \times \\
&\times \delta_{\alpha\mu_i} \tilde{\delta}(\mu(T_1), \mu(T_2)) Z_{\mu(S\cup T\setminus\{T_1\cup T_2\})} .
\end{aligned} \tag{C.1}$$

We can write this deformation of the Leibniz rule in terms of operators acting on $Z_{\mu(S)} \otimes Z_{\mu(T)}$ as follows

$$\begin{aligned}
&Q_\alpha \left(Z_{\mu(S)} * Z_{\mu(T)}\right) - \left(Q_\alpha Z_{\mu(S)}\right) * Z_{\mu(T)} - Z_{\mu(S)} * \left(Q_\alpha Z_{\mu(T)}\right) = \\
&= -\frac{1}{R^2} \sum_{k=0}^{n} \sum_{a_1\cdots a_k} \frac{2k}{n^{2k}} \frac{n!}{k!(n-k)!} m_2^c (m_2^c \otimes 1) \times \\
&\times (1 \otimes \delta_{a_1} \cdots \delta_{a_k} \otimes \delta_{a_1} \cdots \delta_{a_k}) \left(Z_\alpha \otimes Z_{\mu(S)} \otimes Z_{\mu(T)}\right) +
\end{aligned}$$

$$+\frac{1}{R^2}\sum_{k=0}^{n}\sum_{b_1\cdots b_k}\frac{1}{n^{2k}}\frac{n!}{k!(n-k)!}\left(\frac{-2k}{n}+\frac{|S|k}{n^2}\right)\times$$
$$\times\, m_2^c(m_2^c\otimes 1)(1\otimes\delta_{b_1}\cdots\delta_{b_k}\otimes\delta_{b_1}\cdots\delta_{b_k})(\delta_a\otimes 1\otimes\delta_a)(Z_\alpha\otimes Z_{\mu(S)}\otimes Z_{\mu(T)})+$$
$$+\frac{1}{R^2}\sum_{k=0}^{n}\sum_{b_1\cdots b_k}\frac{1}{n^{2k}}\frac{n!}{k!(n-k)!}\left(\frac{-2k}{n}+\frac{|T|k}{n^2}\right)\times$$
$$\times\, m_2^c(m_2^c\otimes 1)(1\otimes\delta_{b_1}\cdots\delta_{b_k}\otimes\delta_{b_1}\cdots\delta_{b_k})\,(\delta_a\otimes\delta_a\otimes 1)\,(Z_\alpha\otimes Z_{\mu(S)}\otimes Z_{\mu(T)})\,. \quad (C.2)$$

Now we can convert the classical products m_2^c into expressions involving m_2^* using the appendix A.

$$Q_\alpha\left(Z_{\mu(S)}*Z_{\mu(T)}\right)-\left(Q_\alpha Z_{\mu(S)}\right)*Z_{\mu(T)}-Z_{\mu(S)}*\left(Q_\alpha Z_{\mu(T)}\right)=$$
$$=-\frac{1}{R^2}\sum_{k=0}^{n}\sum_{a_1\cdots a_l}\sum_{l=0}^{n}\sum_{l=0}^{n}\sum_{b_1\cdots b_l=1}^{2k+1}\frac{2k}{n^{2k}}\frac{n!}{k!(n-k)!}\frac{\tilde f_l(n)}{l!}\times$$
$$\times\; m_2^*(\delta_{c_1}\cdots\delta_{c_l}\otimes\delta_{c_1}\cdots\delta_{c_l})[\;(m_2^*\otimes 1)+\tilde f_1(n)(m_2^*\otimes 1)(\delta_{b_1}\otimes\delta_{b_1}\otimes 1)]\times$$
$$\times\,(1\otimes\delta_{a_1}\cdots\delta_{a_k}\otimes\delta_{a_1}\cdots\delta_{a_k})(Z_\alpha\otimes Z_{\mu(S)}\otimes Z_{\mu(T)})+$$
$$+\frac{1}{R^2}\sum_{k=0}^{n}\sum_{b_1\cdots b_k}\sum_{l=0}^{n}\sum_{c_1\cdots c_l}\frac{1}{n^{2k}}\frac{n!}{k!(n-k)!}\left(\frac{-2k}{n}+\frac{|S|k}{n^2}\right)\frac{\tilde f_l(n)}{l!}\times$$
$$\times\, m_2^*(\delta_{c_1}\cdots\delta_{c_l}\otimes\delta_{c_1}\cdots\delta_{c_l})(m_2^*\otimes 1)\times$$
$$\times\,(1\otimes\delta_{b_1}\cdots\delta_{b_k}\otimes\delta_{b_1}\cdots\delta_{b_k})(\delta_a\otimes 1\otimes\delta_a)(Z_\alpha\otimes Z_{\mu(S)}\otimes Z_{\mu(T)})+$$
$$+\frac{1}{R^2}\sum_{k=0}^{n}\sum_{b_1\cdots b_k}\sum_{l=0}^{n}\sum_{c_1\cdots c_l}\frac{1}{n^{2k}}\frac{n!}{k!(n-k)!}\left(\frac{-2k}{n}+\frac{|T|k}{n^2}\right)\frac{\tilde f_l(n)}{l!}\times$$
$$\times\, m_2^*(\delta_{c_1}\cdots\delta_{c_l}\otimes\delta_{c_1}\cdots\delta_{c_l})(m_2^*\otimes 1)\times$$
$$\times\,(1\otimes\delta_{b_1}\cdots\delta_{b_k}\otimes\delta_{b_1}\cdots\delta_{b_k})(\delta_a\otimes\delta_a\otimes 1)\left(Z_\alpha\otimes Z_{\mu(S)}\otimes Z_{\mu(T)}\right). \quad (C.3)$$

Since an arbitrary element of $\mathcal{B}_n^*(R^{2k+1})$ can be expanded in terms of $Z_{\mu(S)}$ we can replace $Z_{\mu(S)}$ and $Z_{\mu(T)}$ above by arbitrary elements Φ_1,Φ_2 of the algebra.

D. m_2^c in terms of m_2

We have found above that expressing the non-associativity of m_2^* in terms of operators and finding the deformed Leibniz rule for Q_α with respect to m_2^* are conveniently done by having a formula for the classical product in terms of m_2^*. In this paper we started with m_2 and found that if we twist it into m_2^* we get another commutative, non-associative product with nicer properties, notably one on which δ_α acts as a derivation. If one chooses to work with m_2, it is convenient to have a formula for m_2^c in terms of m_2. Let us first note that the defining formula for m_2 (2.13) can be expressed as a formula for m_2 in terms of m_2^c

$$m_2=\sum_{k=0}^{n}\sum_{a_1\cdots a_k} m_2^c\cdot(\delta_{a_1}\cdots\delta_{a_k}\otimes\delta_{a_1}\cdots\delta_{a_k})\frac{(n-D+2k)!}{k!(n-D+k)!}\,. \quad (D.1)$$

D acting on an element of $\mathcal{B}_n \otimes \mathcal{B}_n$ is just the sum of the degrees of the two elements. For example on $Z_{\mu(S)} \otimes Z_{\mu(T)}$ it is $|S| + |T|$. Let us define $g_k(n, D) = \frac{(n-D+2k)!}{k!(n-D+k)!}$ Inverting (D.1) to give m_2^c in terms of m_2^c to find the coefficients $\tilde{g}_k(n, D)$ which appear in the inverse formula

$$m_2^c = \sum_{k=0}^{n} \sum_{a_1 \cdots a_k} m_2.(\delta_{a_1} \cdots \delta_{a_k} \otimes \delta_{a_1} \cdots \delta_{a_k}) \tilde{g}_k(n, D) \tag{D.2}$$

leads to sums of the form

$$\tilde{g}_k(n, D) = \sum_{l=0}^{n} \sum_{a_1 \cdots a_l} g_{a_1}(n, D) g_{a_2}(n, D - 2a_1) \cdots g_{a_l}(n, D - 2a_1 - 2a_2 \cdots 2a_{l-1}) \frac{k!}{a_1! a_2! \cdots a_l!} . \tag{D.3}$$

There is a sum over l, and for each l a sum over choices of ordered sets of positive integers $a_1, a_2 \cdots a_l$ which satisfy $a_1 + a_2 + \cdots a_l = k$. The first few examples of $\tilde{g}_1, \tilde{g}_2, \tilde{g}_3, \tilde{g}_4, \tilde{g}_5$ were done by using Maple (some of them are also easily checked by hand). They all agree with a simple general formula which is

$$\tilde{g}_k(n, D) = \frac{(-1)^k}{n^{2k}} (n - D + 2k) \frac{(n - D + k - 1)!}{(n - D)!} . \tag{D.4}$$

An analytic proof for general k would be desirable. This $\tilde{g}_k$ can be used to write the associator and deformed Leibniz rules involving the product m_2.

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